UNIVERSALITY in TURBULENCE: an EXACTLY SOLUBLE MODEL[1]

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Abstract

The present note contains the text of lectures discussing the problem of universality in fully developed turbulence. After a brief description of Kolmogorov's 1941 scaling theory of turbulence and a comparison between the statistical approach to turbulence and field theory, we discuss a simple model of turbulent advection which is exactly soluble but whose exact solution is still difficult to analyze. The model exhibits a restricted universality. Its correlation functions contain terms with universal but anomalous scaling but with non-universal amplitudes typically diverging with the growing size of the system. Strict universality applies only after such terms have been removed leaving renormalized correlators with normal scaling. We expect that the necessity of such an infrared renormalization is a characteristic feature of universality in turbulence.

1 Introduction

One of the basic and most successful ideas in theoretical physics has been that of universality. It states that many systems with large or infinite number of degrees of freedom in certain asymptotic regimes display similar behaviors falling into general types. Such a situation has been encountered

in statistical mechanics of 2^{nd} order phase transitions where the universality applies to the long distance asymptotics of correlation functions characterized e.g. by scaling exponents, independent of numerous microscopic details of the systems [1],

in field theory where the universality implies the cutoff independence of the effective low energy description [1],

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in the theory of 1^{st} order phase transitions exhibiting universal shapes of critical droplets [2],

in many-body condensed matter where universality limits possible behaviors of superconductors [3] or Hall fluids [4],

in dynamical systems where it describes types of behavior of families of maps under iteration [5],

in non-linear PDE's where it determines space and/or time asymptotics of solutions [6].

The basic idea behind universality of certain behaviors of systems with many degrees of freedom is that such behaviors are governed by few relevant degrees of freedom with simple dynamics. Those may be often exhibited using renormalization group transformations wiping out the irrelevant details of the system [7][8].

The question addressed in the present lectures is to what extent such a picture applies to the fully developed hydrodynamic turbulence. It should be mentioned at the very beginning that the problem is still wide open although the first theory arguing for universality in turbulence, due to Kolmogorov [9], dates from 1941. In the 1^{st} lecture, we shall sketch how the main claims of the Kolmogorov theory relate to the cascade picture of energy transfer in a turbulent flow. In the 2^{nd} lecture, we shall briefly present the functional-integral approach to the statistical theory of turbulence governed by the Navier-Stokes equation trying to stress the similarities and the differences with field theory. In the 3^{rd} and the 4^{th} lecture, we shall discuss a simple model of turbulent advection of a scalar quantity, known under the name of passive scalar (PS). The model is a good playground for testing the idea of universality in turbulence since one may obtain closed expressions for the stochastic initial data evolution and for the steady state correlation functions of the scalar. In the absence of external sources, the PS with deterministic initial data undergoes a non-universal diffusion which averaged over initial data distributed in a homogeneous way becomes a universal super-diffusion. In the presence of random sources, the steady state 2-point correlator exhibits an energy cascade of the type expected in the Navier-Stokes turbulence. The exact solution for the higher point correlators are more difficult to analyze. We examine the origin of possible violations of strict universality of these correlators. What emerges is a picture of restricted universality applying to infrared renormalized correlators from which few universal terms multiplied by non-universal coefficients where removed. In Conclusions, we summarize our discussion and point to possible directions of the future research.

The PS model, although introduced long time ago [10], has become recently a subject of intensive study, see [11][12] and also [13] (the latter, for a mathematical work on a simplified version of the model). In the final stages of the work on these notes, we have received preprints [14][15] and [16]. The analysis of the first two has large overlaps with ours although we disagree with some conclusions of [14] and differ in some interpretations from [15].

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2 Energy cascade and Kolmogorov scaling

The evolution of the local velocity field $\mathbf{v}(t,\mathbf{x})$ of the incompressible fluid at points $\mathbf{x}$ of the 3-dimensional space is described by the Navier-Stokes (NS) equation

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\Delta\mathbf{v} \,=\, -\nabla p + \mathbf{f} \tag{2.1}$$

supplemented with the incompressibility condition

$$\nabla\cdot\mathbf{v} \,=\, 0\ . \tag{2.2}$$

The presence of the dissipative term multiplied by the viscosity ν of the fluid (with the dimension $\frac{length^2}{time}$) distinguishes the NS equation from the Euler equation. $\mathbf{f}(t,\mathbf{x})$ denotes the external force. $p(t,\mathbf{x})$ is the pressure (divided by the constant density) and may be eliminated from both equations:

$$p(t,\mathbf{x}) \,=\, \frac{1}{4\pi}\int \frac{\nabla\cdot[\mathbf{f}(t,\mathbf{y})-(\mathbf{v}(t,\mathbf{y})\cdot\nabla)\mathbf{v}(t,\mathbf{y})]}{|\mathbf{x}-\mathbf{y}|}\,d^3\mathbf{y}\ . \tag{2.3}$$

If the flow takes place in a finite volume, boundary conditions should be specified. The size of the volume introduces an "integral scale" L into the problem and we shall assume that the external force $\mathbf{f}$ acts only at length scales comparable with L. Alternatively, we may consider flows in infinite space excited by external force with Fourier components concentrated in wavenumbers $\mathbf{k}$ with $k\equiv|\mathbf{k}|\leq L^{-1}$. Under the rescaling

$$\begin{aligned}\widetilde{\mathbf{v}}(t,\mathbf{x}) &= \sigma\,\mathbf{v}(\tau t, s\mathbf{x})\ ,\\ \widetilde{\mathbf{f}}(t,\mathbf{x}) &= \sigma\tau\,\mathbf{f}(\tau t, s\mathbf{x})\ ,\\ \widetilde{p}(t,\mathbf{x}) &= \sigma\tau s^{-1}p(\tau t, s\mathbf{x})\end{aligned} \tag{2.4}$$

with $\sigma s/\tau\,=\,1$, the viscosity ν is replaced in Eq. (2.1) by $\widetilde{\nu}\,=\,\tau s^{-2}\nu$. It is then convenient to introduce a dimensionless quantity, the Reynolds number

$$R \,=\, \frac{V\,L}{\nu}\ , \tag{2.5}$$

where V stands for the typical value of the velocity differences over the integral scale L. R is invariant under the rescalings (2.4).

The basic phenomenological observation is that the hydrodynamic flows have very different behavior for small values of R (of order 1, say) where flows are regular (laminar) and large values of R (of order 10^3 and more) where the flows are very chaotic (turbulent), with complicated phenomena occurring at intermediate values of R. Here, we shall be interested in the regime of fully developed turbulence with very large Reynolds numbers, ideally in the limit $R\to\infty$. It is sensible to use a statistical description of the flows in this regime. The stochasticity may be introduced into the description by considering random external force $\mathbf{f}$ or studying the evolution of random initial data or both. In a statistical description, the basic objects to look at will be the velocity correlation functions given by the expectation values of products of components of velocities. We shall mostly look at equal time correlators

$$<\prod_{n=1}^{N} v^{i_n}(t,\mathbf{x}_n)> \tag{2.6}$$

which, in the stationary state of turbulence sustained by steady external forces, should be time independent. Besides, we may expect that, far from the fluid boundaries, the correlators (2.6) are translationally and rotationally invariant, the property that expresses the homogeneity and isotropy of the fully developed turbulence. The most commonly studied velocity correlators are the so called structure functions

$$S_N(x) \;=<\,(\mathbf{v}(\mathbf{x})-\mathbf{v}(0))^N\,>\ \ . \tag{2.7}$$

One of the simplest consequences of the NS equation (2.1) may be obtained by taking its scalar product with $\mathbf{v}$ and integrating the result over the space. These results in the relation

$$\tfrac{d}{dt}\,\tfrac{1}{2}\int\mathbf{v}^2\;=\;-\tfrac{\nu}{2}\int(\nabla\mathbf{v})^2\,+\,\int\mathbf{v}\cdot\mathbf{f} \tag{2.8}$$

which expresses the energy balance: the time derivative of energy on the right hand side is equal to the difference of the injection rate $\int\mathbf{f}\cdot\mathbf{v}$ and the dissipation rate $\frac{\nu}{2}\int(\nabla\mathbf{v})^2$ of energy. Taking averages in the stationary state, we obtain

$$<\mathbf{v}\cdot\mathbf{f}>\,=\,<\tfrac{\nu}{2}(\nabla\mathbf{v})^2>\,\equiv\,\epsilon\ , \tag{2.9}$$

i.e. the equality of the (intensive) mean injection and the mean dissipation rates of energy. How is this energetic balance realized over different modes of the fluid motion? The energy injection takes place at the distances of order of the integral scale by induction of big scale L eddies. According to the picture of the turbulent flow proposed in 1922 by Richardson [17], the big eddies induce smaller eddies which, in turn, induce still smaller eddies and so on transferring energy from large to small distance scales. This process should not lead to a loss of energy until sufficiently small distance scales, say, smaller than η, are reached. On scales smaller than η, the dissipative term $\nu\,\Delta\mathbf{v}$ of the NS equation becomes important. One may formulate a more quantitative version of this picture. We shall follow here the discussion of the treatise [18] which is an excellent reference to the problems of well developed turbulence. The idea is to localize the velocity field in the Fourier space. For $\widehat{\mathbf{v}}(\mathbf{k})\equiv\int\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{v}(\mathbf{x})\,d^3\mathbf{x}$ define

$$\mathbf{v}_{\leq K}(\mathbf{x})\;=\;\int\limits_{|k|\leq K}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,\widehat{\mathbf{v}}(\mathbf{k})\,\tfrac{d^3\mathbf{k}}{(2\pi)^3}\ . \tag{2.10}$$

A more detailed version of the energy balance equation (2.8) reads

$$\tfrac{d}{dt}\,\tfrac{1}{2}\int\mathbf{v}_{\leq K}^{\,2}\;=\;-\tfrac{\nu}{2}\int(\nabla\mathbf{v}_{\leq K})^2\,+\,\int\mathbf{v}_{\leq K}\cdot\mathbf{f}\,-\,\int\Pi_K\ , \tag{2.11}$$

where Π_K is the density of energy flux out of the wave numbers with $|\mathbf{k}|\leq K$. An explicit expression for Π_K may be obtained from the NS equation (2.1) and may be argued to involve mainly $\widehat{\mathbf{v}}(\mathbf{k})$ with $|\mathbf{k}|\cong K$, i.e. to be aproximately local in the wavenumber space. Taking the averages in the stationary state, we obtain the relation

$$<\mathbf{v}_{\leq K}\cdot\mathbf{f}>\,=\,<\tfrac{\nu}{2}\int(\nabla\mathbf{v}_{\leq K})^2>\,+\,<\Pi_K> \tag{2.12}$$

which states that the mean injection rate of energy into the modes with $|\mathbf{k}|\leq K$ is equal to the mean rate of dissipation of energy $\epsilon_{\leq K}\;\equiv<\frac{\nu}{2}\int(\nabla\mathbf{v}_{\leq K})^2>$ in those modes plus the mean flux of energy out of modes with $|\mathbf{k}|\leq K$. According to the cascade picture of the energy transfer, the dissipation rate ϵ_K is essentially zero for $K\ll\eta^{-1}$. Since the injection takes place only around the integral scale, the change of the mean injection

rate $\varphi_{\le K} \equiv < \mathbf{v}_{\le K} \cdot \mathbf{f} >$ should be negligeable for $K \gg L^{-1}$ so that $\varphi_{\le K}$ should be equal to its $K = \infty$ value ϵ for such K. It follows from Eq. (2.12) that in the so called **inertial range** of scales $L^{-1} \ll K \ll \eta^{-1}$, the mean energy flux in the wavenumber space $\pi_K \equiv < \Pi_K >$ is constant. This is an important quantitative consequence of the cascade picture of developed turbulence which, however, is not easily verifiable directly.

Kolmogorov went further, postulating that in the inertial range $\eta \ll x \ll L$, the velocity structure functions may be expressed solely as universal functions of the mean dissipation rate ϵ and the distance x, due to the locality and self-similarity of the cascade. The shape of the structure functions is then dictated by the dimensional considerations. One obtains

$$S_N(x) \;=\; C_N\, \epsilon^{N/3}\, x^{N/3} \tag{2.13}$$

with universal coefficients C_N (the right hand side is the only function of ϵ and x with dimension $(\frac{length}{time})^N$). Indeed, this behavior is consistent with the cascade picture. With the typical velocity of size l eddies $v_l \sim S_N(l)^{1/N} \sim \epsilon^{1/3} l^{1/3}$, their typical energy density $\frac{1}{2} v_l^2 \sim \epsilon^{2/3} l^{2/3}$ and their typical turnover time $t_l \sim l/v_l \sim \epsilon^{-1/3} l^{2/3}$, one may estimate the energy flux from scales longer than l to those shorter than l as $\sim v_l^2/t_l \sim \epsilon$ with no scale dependence.

The scaling (2.13) of the 2$^{\rm nd}$ structure function translates under the Fourier transform to the relation

$$\tfrac{1}{4\pi k^2}\, E(k) \;\equiv < \mathbf{v}(\mathbf{k}) \cdot \mathbf{v}(-\mathbf{k}) > = \int \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\, S_2(x)\, d^3\mathbf{x} \;\sim\; \epsilon^{2/3} k^{-11/3} \tag{2.14}$$

so that the Kolmogorov theory prediction for the energy spectrum $E(k)$ is

$$E(k) \;\sim\; \epsilon^{2/3}\, k^{-5/3}\;. \tag{2.15}$$

The theory allows also to locate the scale η at which the dissipative effects become important, called usually the Kolmogorov scale. Estimating the dissipation rate on scale l as $\sim \nu(\frac{v_l}{l})^2$, η should correspond to the scale at which $\nu(\frac{v_\eta}{\eta})^2 \cong \epsilon$. Using the relations

$$v_\eta \sim \epsilon^{1/3} \eta^{1/3}\;, \qquad\quad v_L \sim \epsilon^{1/3} L^{1/3}\;, \tag{2.16}$$

we infer that

$$\eta \sim (\tfrac{\nu}{v_L L})^{3/4} L \;=\; R^{-3/4} L\;. \tag{2.17}$$

As should be expected, the Kolmogorov scale decreases with growing Reynolds numbers, i.e. with decreasing viscosity. In the limit $R \to \infty$, the inertial range should invade all short scales. It does not mean, however, that the behavior in that range may be described by the Euler equation: the presence of even very small viscosity is essential for the existence of the stationary state of the turbulence by providing a mechanism for removal from the system of the energy injected by external forces.

The verifications of the Kolmogorov theory is made difficult by the fact that both in experiments in the atmosphere, in wind tunnels or in liquid flows and in numerical simulations it is hard to obtain inertial ranges extending over many orders of magnitude necessary to extract the characteristic exponents of the structure functions. Nevertheless, it seems that the behavior of low structure functions well agrees with Kolmogorov's

predictions whereas for higher structure functions with $N \sim 10$ one observes values of exponents lower than predicted (by $\sim 10\%$ or more). A multitude of more or less *ad hoc* mechanisms has been proposed to explain the possible departures from the Kolmogorov scaling for the higher structure functions. The underlying idea is that of intermittency limiting the turbulent activity to a subset of temporal and spatial modes. We shall not discuss the intermittent models here referring the interested reader to [18].

3 Functional methods in turbulence

The Kolmogorov theory has a phenomenological character. It is strongly related to the cascade picture of turbulence which was postulated essentially independently of the NS equations. It may be compared to the mean-field approach to critical phenomena which leads to correct results only in limited situations and which, in general, has to be replaced by a more sophisticated theory taking into account the role of statistical fluctuations. It is then essential to try to build a theory of fully developed turbulence starting from the basic equations.

For any functional evolution equation of the type

$$\partial_t \Phi = -\mathcal{F}(\Phi) + F \,, \tag{3.1}$$

where F is a functional Gaussian process, a Martin-Siggia-Rose (MSR) formalism [19] permits to obtain formal functional integral expressions for the expectation values of functionals $\mathcal{A}(\Phi)$ in the stationary state. This is done as follows. We rewrite

$$< \mathcal{A}(\Phi) > = \frac{1}{\mathcal{N}} \int \mathcal{A}(\Phi)\ \delta(\partial_t \Phi + \mathcal{F}(\Phi) - F)\ \det(\partial_t + \tfrac{\delta \mathcal{F}(\Phi)}{\delta \Phi})\ \mathrm{e}^{-\frac{1}{2}(F, \mathcal{C}^{-1} F)}\ D\Phi\, DF, \tag{3.2}$$

where $\mathcal{C}$ is the covariance operator of the Gaussian process F and $\mathcal{N}$ is the normalizing factor given by a similar integral with $\mathcal{A} = 1$. The role of the Φ integral is to express Φ as a solution of the evolution equation (3.1) (with a fixed initial condition). If the functional $\mathcal{F}(\Phi)$ is local in time then, formally,

$$\det(\partial_t + \tfrac{\delta \mathcal{F}(\Phi)}{\delta \Phi}) = \det(\partial_t)\ \det(1 + \partial_t^{-1} \tfrac{\delta \mathcal{F}(\Phi)}{\delta \Phi}) = \det(\partial_t)\ \mathrm{e}^{\frac{1}{2}\, \mathbf{tr}\, \delta \mathcal{F}(\Phi)/\delta \Phi} \tag{3.3}$$

since $\partial_t^{-1}(t, t') = \theta(t - t')$ and $\delta \mathcal{F}(\Phi)/\delta \Phi$ is proportional to $\delta(t - t')$. Alternatively, one may use the (Berezin) functional integral over anticommuting fields to exponentiate the determinant. Rewriting also the delta function $\delta(\partial_t \Phi + \mathcal{F}(\Phi) - F)$ as an oscillatory integral, we obtain

$$< \mathcal{A}(\Phi) > = \frac{1}{\mathcal{N}} \int \mathcal{A}(\Phi)\ \mathrm{e}^{i\,(A\,,\, \partial_t \Phi + \mathcal{F}(\Phi) - F) + \frac{1}{2}\mathbf{tr}\, \delta \mathcal{F}(\Phi)/\delta \Phi - \frac{1}{2}(F, \mathcal{C}^{-1} F)}\ DA\, D\Phi\, DF\ . \tag{3.4}$$

Finally, the Gaussian integral over F leads to the relation

$$< \mathcal{A}(\Phi) > = \frac{1}{\mathcal{N}} \int \mathcal{A}(\Phi)\ \mathrm{e}^{i\,(A\,,\, \partial_t \Phi + \mathcal{F}(\Phi)) + \frac{1}{2}\mathbf{tr}\, \delta \mathcal{F}(\Phi)/\delta \Phi - \frac{1}{2}(A, \mathcal{C} A)}\ DA\, D\Phi\ . \tag{3.5}$$

The MSR formalism applies to the Langevin equation describing the approach to equilibrium in statistical mechanics and Euclidean field theory. In this case, the functional $\mathcal{F}(\Phi)$ appearing in the evolution equation (3.1) is of the gradient type:

$$\mathcal{F}(\Phi) = \frac{\delta S(\Phi)}{\delta \Phi} \tag{3.6}$$

and the Gaussian process F is the white noise, i.e. its covariance $\mathcal{C}$ is the identity operator. For example, for the scalar ϕ^4 theory,

$$S(\Phi) \,=\, \int \left(\tfrac{1}{2}\nabla\Phi)^2 + \tfrac{1}{2}m^2\Phi^2 + \lambda\Phi^4\right) dt\, d^d\mathbf{x}\ . \tag{3.7}$$

Perturbative treatment of the functional integral (3.5) based on expanding the exponential of the terms of order higher then 2 in Φ and A into a power series gives rise to the perturbative expansion for the correlation functions. The latter reduces for the equal-time correlators to the perturbative expansion for the equilibrium expectations in the Gibbs measure $\frac{1}{\mathcal{N}}\,\mathrm{e}^{-S(\phi)}\,D\phi$ with ϕ as Φ but without the time dependence.

One may rewrite the NS equations (2.1) and (2.2) as a single evolution equation

$$\partial_t\mathbf{v} \,=\, -P\,(\mathbf{v}\cdot\nabla)\mathbf{v} \,+\, \nu\,\Delta\mathbf{v} \,+\, P\mathbf{f} \tag{3.8}$$

in the space of divergence-free vector fields, where P stands for the orthogonal projection on such fields. Taking the external force $\mathbf{f}$ random Gaussian, we end up in the setup formally resembling the Langevin dynamics. The important differences between the latter and Eq. (3.8) are, however, that, first, the term $P\,(\mathbf{v}\cdot\nabla)\mathbf{v}$ is not of the gradient type and, second, that we want to look at the noise $\mathbf{f}$ with the Fourier components concentrated in low wavenumbers, i.e. with the kernel of the covariance in the position space close to a constant as opposed to the delta function kernel for the Langevin equation. As a result, the role of the noise in the NS equation is to inject at small wavenumbers the energy which is then transferred to larger wavenumbers by an essentially deterministic process leading to the non-zero energy flux in the stationary state whereas in the Langevin dynamics, the noise plays an essential role on all scales resulting in the thermal state with no non-zero energy fluxes between scales.

It is sometimes pointed [20] that one encounters non-vanishing momentum-space fluxes also in the "equilibrium" field theory in the presence of anomalies. For the sake of illustration, let us consider the chiral anomaly in two space-time dimensions. The chiral charge density of the Dirac field is

$$J^{50}(t,x) = \bar{\psi}(t,x)\gamma^5\gamma^0\psi(t,x)\ . \tag{3.9}$$

The anomaly calculation shows that in the external abelian gauge field in two space-time dimensions

$$\tfrac{d}{dt} < \textstyle\int J^{50}(t,x)\,dx >_{\leq K} \,=\, \tfrac{1}{2\pi}\textstyle\int \epsilon^{\mu\nu}\,F_{\mu\nu}(t,x)\,dx\ , \tag{3.10}$$

and the right hand side is K independent. Here the subscript $\leq K$ denotes a gauge-invariant (e.g. Pauli-Villars) subtraction at momentum-scale K. In other words, the flux of the chiral charge out of the modes with momenta $\leq K$ is constant in K. It is not clear what lessons may be drawn from this analogy for understanding the inertial range of the fully developed turbulence.

Applying the MSR formalism to Eq. (3.8), we obtain for the velocity correlation functions the expression

$$< \prod_{n=1}^{N} v^{i_n}(t_n,\mathbf{x}_n) >= \frac{1}{\mathcal{N}}\int \prod_{n=1}^{N} v^{i_n}(t_n,\mathbf{x}_n)\ \mathrm{e}^{\,i\,(\mathbf{a},\,\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}-\nu\,\Delta\mathbf{v})-\frac{1}{2}(\mathbf{a},\mathcal{C}\mathbf{a})}\ D\mathbf{a}\,D\mathbf{v}\,, \tag{3.11}$$

where the $\mathbf{a}$- and $\mathbf{v}$-integrals are over divergence-free vector fields (tr $\frac{\delta(\mathbf{v}\cdot\nabla)\mathbf{v}}{\delta\mathbf{v}}$ drops out as it vanishes because of symmetry reasons). Separating the expression under the exponential

into the quadratic part $i\,(\mathbf{a},\,\partial_t\mathbf{v}-\nu\,\Delta\mathbf{v})-\frac{1}{2}(\mathbf{a},\,\mathcal{C}\mathbf{a})$ and the cubic one $i\,(\mathbf{a},\,(\mathbf{v}\cdot\nabla)\mathbf{v})$ and expanding the exponential of the latter into the power series, we may generate the perturbation expansion for the velocity correlators, just like for the correlation functions of the Langevin evolution. The terms of the expansion may be represented by Feynman diagrams with the propagators read off the quadratic part under the exponential and the vertices off the cubic part. A closer look into the perturbation expansion generated this way shows that it is plagued by infrared and ultraviolet divergences (the latter in the limit of vanishing ν). Various resummation techniques, mostly inspired by approaches used in field theory, were tried with the aim to improve the convergence, without complete success. Let us mention here another approach, originally labeled quasi-Lagrangian, which seems more promising. This approach has been developed over years by the Novosibirsk and more recently the Weizmann Institute schools [21][22][23].

The idea consists of describing the turbulent flow in the frame moving with one of its points. Suppose that, for given velocity field $\mathbf{v}(t,\mathbf{x})$, $\mathbf{x}(t)$ is the solution of the ODE

$$\frac{d\mathbf{x}}{dt}=\mathbf{v}(t,\mathbf{x}) \tag{3.12}$$

with the initial condition $\mathbf{x}(t_0)=\mathbf{x}_0$. $\mathbf{x}(t)$ is the trajectory of the material point of the fluid located at $\mathbf{x}_0$ at time t_0. The velocity field, force and pressure in the frame moving with the fluid point are

$$\mathbf{v}'(t,\mathbf{x}) \;=\; \mathbf{v}(t,\mathbf{x}+\mathbf{x}(t))\;, \tag{3.13}$$
$$\mathbf{f}'(t,\mathbf{x}) \;=\; \mathbf{f}(t,\mathbf{x}+\mathbf{x}(t))\;, \tag{3.14}$$
$$p'(t,\mathbf{x}) \;=\; p(t,\mathbf{x}+\mathbf{x}(t))\;, \tag{3.15}$$

and they satisfy the equations following from the NS equations for $\mathbf{v}$:

$$\partial_t\mathbf{v}'+((\mathbf{v}'-\mathbf{v}'_0)\cdot\nabla)\mathbf{v}'-\nu\Delta\mathbf{v}' \;=\; -\nabla p'+\mathbf{f}'\;, \tag{3.16}$$
$$\nabla\cdot\mathbf{v}' \;=\; 0\;, \tag{3.17}$$

where $\mathbf{v}'_0(t)\equiv\mathbf{v}'_0(t,0)$. Eqs. (3.16) and (3.17) may be again written as a single evolution equation in the space of divergence-free vector fields:

$$\partial_t\mathbf{v}' \;=\; -P\,((\mathbf{v}'-\mathbf{v}'_0)\cdot\nabla)\mathbf{v}'\,+\,\nu\,\Delta\mathbf{v}'\,+\,P\mathbf{f}' \tag{3.18}$$

to which the MSR formalism may be applied if we assume that $\mathbf{f}'$ is a Gaussian random field. This is not the same as a similar assumption about $\mathbf{f}$ but as long as the Fourier components of the force are concentrated in small wavenumbers, the details of the force distribution should not matter in the inertial range. In such a situation, one may expect that the equal-time velocity correlators in the stationary state of the evolution equation (3.18) coincide in the inertial range with those corresponding to the equation (3.8). The perturbation expansion generated by the MSR technique for the $\mathbf{v}'$-correlators has the same propagators but modified vertices which break the translation invariance and make this way the quasi-Lagrangian expansion more difficult to analyze. Nevertheless it seems that the quasi-Lagrangian approach leads to drastic improvement in convergence of the (Schwinger-Dyson resummation) of the perturbative expansion [21][23] and it is possible that it provides an important clue to the understanding of the statistical properties of the turbulent flow.

4 Passive scalar

4.1 Definition of the model

In view of the difficulties encountered by the theory of the Navier-Stokes turbulence, it may be useful to search for simpler systems exhibiting some of the aspects of full-fledged turbulence but easier to control. Such a system seems to be provided by a model of passive advection in a random velocity field $\mathbf{v}(t,\mathbf{x})$ of a scalar quantity T whose density $T(t,\mathbf{x})$ satisfies the equation

$$\partial_t T + (\mathbf{v}\cdot\nabla)T - \nu\Delta T = f\ , \tag{4.1}$$

where now ν denotes the molecular diffusivity of the scalar T and $f(t,\mathbf{x})$ describes the external sources. In the ideal situation, we should consider $\mathbf{v}$ as the velocity of the NS turbulent flow, but we shall, instead, assume that $\mathbf{v}(t,\mathbf{x})$ is a centered Gaussian field with the covariance

$$< v^i(t,\mathbf{x})\, v^j(t',\mathbf{x}') >= \delta(t-t')\ D^{ij}(\mathbf{x}-\mathbf{x}')\ , \tag{4.2}$$

i.e. white noise in time. The spatial part of the covariance will be taken, for concreteness, as

$$D^{ij}(\mathbf{x}) \,=\, D_0 \int \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\, (\mathbf{k}^2+m^2)^{-(3+\kappa)/2}\, (\delta^{ij}-k^ik^j/\mathbf{k}^2)\, \tfrac{d^3\mathbf{k}}{(2\pi)^3}\ , \tag{4.3}$$

where the transverse projector in the Fourier space ensures the incompressibility of $\mathbf{v}$. Small m^2 should be viewed as an infrared cutoff making the integral convergent for $0<\kappa<2$. Note that the dimension of D_0 is $\frac{length^{2-\kappa}}{time}$. For the 2$^{\rm nd}$ velocity structure function we obtain

$$< (\mathbf{v}(t,\mathbf{x})-\mathbf{v}(t,0))^2 > \sim\ D^{ii}(0) - D^{ii}(\mathbf{x}) \equiv \widetilde{D}^{ii}(\mathbf{x}) \tag{4.4}$$

which, for $0<\kappa<2$, has the $m\to 0$ limit proportional to $|\mathbf{x}|^\kappa$. i.e. growing with the distance. More exactly,

$$D^{ij}(0) \,=\, \tfrac{16\,\Gamma(\kappa/2)}{3\pi^{3/2}\,\Gamma(\kappa/2)}\,\delta^{ij}\, D_0\, m^{-\kappa}\ , \tag{4.5}$$

i.e. it diverges with $m\to 0$, but

$$\lim_{m\to 0}\ \widetilde{D}^{ij}(\mathbf{x}) \,=\, D_1\,\Big((2+\kappa)\,\delta^{ij}\,|\mathbf{x}|^\kappa \,-\, \kappa\, x^i x^j\, |\mathbf{x}|^{\kappa-2}\Big) \tag{4.6}$$

and it is a homogeneous function of $\mathbf{x}$. $D_1 \equiv \frac{\Gamma((2-\kappa)/2)}{2^{2+\kappa}\pi^{3/2}\kappa(3+\kappa)}\, D_0$. Note the singularities at 0 and 2 in the κ-dependence. The Kolmogorov scaling (2.13) corresponds to $\kappa=\frac{2}{3}$ which lies in the interval $0<\kappa<2$ that we shall consider. It should be mentioned, however, that the time-decorrelation, evenness and Gaussian character of the velocity distribution is a rather drastic departure from the statistical properties of the turbulent NS velocities. Nevertheless, the model, dating back to 1949 [10], possesses an interest of its own and has been used to describe a variety of phenomena like poluant or tracer transport and is closely related to models describing forced diffusion through porous media [24].

We shall take the field $f(t,\mathbf{x})$ describing the external sources also random Gaussian, independent of $\mathbf{v}$, with mean zero and covariance

$$< f(t,\mathbf{x})\, f(t',\mathbf{x}') >= \delta(t-t')\ \mathcal{C}(\tfrac{\mathbf{x}-\mathbf{x}'}{L}) \equiv \delta(t-t')\ \mathcal{C}_L(\mathbf{x}-\mathbf{x}') \tag{4.7}$$

where $\mathcal{C}$ is a real positive-definite function from the Schwartz space $\mathcal{S}(\mathbf{R}^3)$ and L is the integral scale. Sometimes, it will be useful to consider also the advection of a complex scalar in which case $\mathcal{C}$ may be a complex positive-definite function. We would like to study the statistical properties of the solutions of Eq. (4.1) in the regime of small ν, small m (which may viewed as the inverse of another integral scale) and large L (in the limit $L \to \infty$, the field f has only the zero wavenumber component). In particular, the universality question for the PS may be formulated as inquiring about the existence of the limit of the correlation functions

$$< \prod_{n=1}^{N} T(t_n, \mathbf{x}_n) > \tag{4.8}$$

in a stationary state of the system when $\nu, m, L^{-1} \to 0$ and about the independence of such a limit of the shape of the source covariance $\mathcal{C}$.

4.2 Exact solution

The big advantage of the PS model over the NS case is that one may obtain closed expressions for the correlation functions of the scalar (the assumed time-decorrelation of the velocity field plays here the crucial role). For 2-point function, this was first done in [25][3]. In this sense the model is exactly soluble. In order to work out its exact solution, let us rewrite the basic evolution equation (4.1) of the scalar in a general form

$$\partial_t T + \beta T + \nu\alpha T = f \ , \tag{4.9}$$

where β is a skew-symmetric random operator ($\beta = \mathbf{v} \cdot \nabla$ in our case) and α is a positive operator ($\alpha = -\Delta$). The β term in the equation is conservative whereas the α one is dissipative (for the energy $\frac{1}{2}\|T\|^2$). For the sake of simplicity, we shall argue in the finite-dimensional case, i.e. when $T(t)$ and $f(t)$ take values in $\mathbf{R}^d$ and $\beta(t)$ and α are, respectively, skew-symmetric and positive symmetric $d \times d$ matrices. For the covariances of independent centered Gaussian processes $\beta(t)$ and $f(t)$, we shall write

$$\begin{aligned} < \beta_{ij}(t)\, \beta_{kl}(s) > \ &= \ \delta(t-s)\, \mathcal{B}_{ij,kl} \ , \\ < f_i(t)\, f_j(s) > \ &= \ \delta(t-s)\, \mathcal{C}_{ij} \ . \end{aligned} \tag{4.10}$$

In finite dimensions, the original stochastic PDE becomes a stochastic ODE which is easier to handle. Let us simplify the matter further by assuming, for a moment, that $\beta(t)$ and $f(t)$ are continuous functions. Then the solution of Eq. (4.9) with the initial condition T_0 at $t = t_0$ takes the well known form

$$T(t) \ = \ R(t,t_0)\, T_0 \ + \int_{t_0}^{t} R(t,s)\, f(s)\, ds \ , \tag{4.11}$$

where $R(t,t_0)$ is given by the ordered exponential ($t \geq t_0$)

$$R(t,t_0) \ = \ \mathcal{T} \ \mathrm{e}^{-\int_{t_0}^{t} (\nu\alpha + \beta(\tau))\, d\tau} \tag{4.12}$$

($\mathcal{T}$ stands for the time ordering). $R(t,t_0)$ may be also expressed by the following limiting formula:

$$R(t,t_0) \ = \ \lim_{Q\to\infty} \ \exp[-\int\limits_{I_Q} (\nu\alpha + \beta(\tau))\, d\tau] \ \exp[-\int\limits_{I_{Q-1}} (\nu\alpha + \beta(\tau))\, d\tau] \ \cdots$$

[3] We thank U. Frisch for explaining to us the 4-point function case

$$\cdots\ \exp[-\int_{I_1}(\nu\alpha+\beta(\tau))\,d\tau\,]\ \equiv\ \lim_{Q\to\infty}\ R_Q(t,t_0)\ , \qquad (4.13)$$

where (I_q) is a time ordered partition of the interval $[t_0,t]$ into subintervals of length $(t-t_0)/Q$.

In the random case, the white noise stochastic processes $\beta(t)$ and $f(t)$ do not have continuous realizations. They need smearing to give genuine random variables. It is easy to see that $\int_I\beta(\tau)\,d\tau\ \equiv\ \beta^I$ where I are intervals of time, are already genuine Gaussian random variables which are mutually independent for non-intersecting or adjacent intervals:

$$<\beta^I_{ij}\,\beta^J_{kl}>=|I\cap J|\ \mathcal{B}_{ij,kl}\ . \qquad (4.14)$$

Let us denote by $\mathcal{B}$ the covariance operator acting on skew-symmetric matrices by

$$(\mathcal{B}\beta)_{ij}\ =\sum_{ijkl}\mathcal{B}_{ij,kl}\,\beta_{kl}\ . \qquad (4.15)$$

Necessarily, $\mathcal{B}\geq 0$. We shall also need below a contracted version B of $\mathcal{B}$ with the matrix elements

$$B_{ij}\ =\ -\tfrac{1}{2}\sum_k\mathcal{B}_{ik,kj}\ . \qquad (4.16)$$

B is a symmetric operator acting in $\mathbf{R}^d$. Note that

$$<((\beta^I)^t\,\beta^I)_{ij}>=-<(\beta^I\,\beta^I)_{ij}>=\ 2|I|\,B_{ij} \qquad (4.17)$$

from which it follows that $B\geq 0$. In the random case with white noise $\beta(t)$, the approximate evolution operators $R_Q(t,t_0)$ under the limit on the right hand side of Eq. (4.13) still make sense for each Q as random (matrix valued) variables and one can show that

Proposition 1.

$$\lim_{Q\to\infty}\ <R_Q(t,t_0)>\ =\ \mathrm{e}^{-(t-t_0)(\nu\alpha+B)}\ \ \equiv\ \ \mathrm{e}^{-(t-t_0)\mathcal{M}_1}\ . \qquad (4.18)$$

In order to indicate how the above result arises, consider the perturbation expansion for $R(t,t_0)$ in powers of β

$$R(t,t_0)\ =\sum_{m=0}^{\infty}(-1)^m\int_{t_0}^{t}d\tau_m\int_{t_0}^{\tau_m}d\tau_{m-1}\cdots\int_{t_0}^{\tau_2}d\tau_1\,\mathrm{e}^{-(t-\tau_m)\,\nu\alpha}\beta(\tau_m)\,\mathrm{e}^{-(\tau_m-\tau_{m-1})\nu\alpha}\,\beta(\tau_{m-1})$$
$$\cdots\ \beta(\tau_1)\,\mathrm{e}^{-(\tau_1-t_0)\nu\alpha}\ . \qquad (4.19)$$

In the white noise expectation value of the right hand side computed with the use of the Wick Theorem, only neighboring $\beta(\tau_m)$ may be paired forcing the respective times τ_m to be equal. One obtains this way the perturbation expansion of $\mathrm{e}^{-(t-t_0)(\nu\alpha+B)}$ in powers of B. Proposition 1 is a rigorous version of this result, not very difficult to prove [26].

Similarly, one may calculate expectations of products of matrix elements of $R_Q(t,t_0)$. We shall use the tensor product notation $R_Q(t,t_0)^{\otimes N}$ as a bookkeeping device for all such

products. Let us denote by $\mathcal{K}$ the symmetric operator on $\mathbf{R}^d \otimes \mathbf{R}^d \cong \mathrm{Mat}_{d\times d}$ built of the coefficients $\mathcal{B}_{ij,kl}$ so that

$$(\mathcal{K}g)_{ik} = \sum_{jl} \mathcal{B}_{ij,kl}\, g_{jl} \; . \tag{4.20}$$

Note the difference between $\mathcal{K}$ and the covariance operator $\mathcal{B}$ of the process $\beta(t)$ defined in Eq. (4.15). In what follows an important role will be played by the N-body symmetric operators $\mathcal{M}_N$ acting in $(\mathbf{R}^d)^{\otimes N} \cong \mathbf{R}^{Nd}$ defined by

$$\mathcal{M}_N \;\equiv\; \sum_{n=1}^{N} (\mathcal{M}_1)_n \;-\; \sum_{1\leq n<n'\leq N} (\mathcal{K})_{n,n'} \;, \tag{4.21}$$

where $\mathcal{M}_1 = \nu\alpha + B$, $(\mathcal{M}_1)_n$ denotes $\mathcal{M}_1$ acting in the n^{th} factor of the tensor product $(\mathbf{R}^d)^{\otimes N}$ and, similarly, $(\mathcal{K})_{n,n'}$ stands for $\mathcal{K}$ acting in the n^{th} and n'^{th} factors of $(\mathbf{R}^d)^{\otimes N}$.

Proposition 2. For $N = 2, 3, \ldots,$

$$\lim_{Q\to\infty} \; < R_Q(t,t_0)^{\otimes N} > \;=\; \mathrm{e}^{-(t-t_0)\,\mathcal{M}_N} \; . \tag{4.22}$$

Again, this result is not surprising if we use the perturbation expansion (4.19) for each $R(t,t_0)$ and note that the white noise expectation may pair now, besides the nearest neighbor $\beta(\tau_m)$ in the perturbative terms of the same $R(t,t_0)$ also $\beta(\tau_m)$ corresponding to different $R(t,t_0)$, resulting in the perturbative expansion of $\mathrm{e}^{-(t-t_0)\mathcal{M}_N}$ in powers of operators B and $\mathcal{K}$. In fact, one may show [26] that $R_Q(t,t_0)$ converge to a limit in all L^p spaces, $p<\infty$, of the white noise β. Naturally, we shall denote the limit by $R(t,t_0)$. $R(t,t_0)$ are contracting operators, they are continuous in t, t_0 in the L^p-norms and satisfy the composition law. We shall take then $T(t)$ given by Eq. (4.11) with the limiting $R(t,t_0)$ as the solution of the stochastic equation (4.9) with initial value T_0.

In the absence of the external sources the time evolution of the equal-time correlators of the scalar T simply reduces to the relations

$$< T(t)^{\otimes N} > \;=\; \mathrm{e}^{-(t-t_0)\mathcal{M}_N}\, T(t_0)^{\otimes N} \;, \tag{4.23}$$

hence it is given by the semigroups $(\mathrm{e}^{-t\,\mathcal{M}_N})_{t\geq 0}$ which may be studied by looking at the Fourier transforms of the semigroups i.e. at the resolvents $(i\omega + \mathcal{M}_N)^{-1}$. In any case, the properties of the time evolution of the statistics of initial data are determined by the spectral properties of the symmetric operators $\mathcal{M}_N$. It is easy to see that the operators $\mathcal{M}_N$ satisfy the inequality

$$\mathcal{M}_N \;\geq\; \nu \sum_{n=1}^{N} (\alpha)_n \; . \tag{4.24}$$

An important consequence of this inequality is the exponential time decay of the correlation functions $< T(t)^{\otimes N} >$ as long as the dissipation is present in the equation (4.9) (recall that we are looking at the finite-dimensional case). The decay rate is given by the lowest eigenvalues $\lambda_{N,0}$ of $\mathcal{M}_N$ and is bounded below by $N\nu\mu_0$ where $\mu_0 > 0$ is the lowest eigenvalue of α. If we remove the dissipative term from the equation by setting $\nu = 0$ then the long time behavior of $< T(t)^{\otimes N} >$ will have a persistent tail since the operators $\mathcal{M}_N$ develop zero modes when ν is taken to zero. Suppose now that the initial

data $T(t_0)$ are also random and independent of $\beta(t)$ for $t \geq t_0$. Averaging Eq. (4.23) over T_0, we obtain

$$< T(t)^{\otimes N} > \,=\, \mathrm{e}^{-(t-t_0)\mathcal{M}_N} < T(t_0)^{\otimes N} > \;. \tag{4.25}$$

A natural problem is the existence of an invariant measure on the space of initial data. It follows from relations (4.24) and (4.25) that there are no invariant measures with finite moments on the space of initial data as long as $\nu > 0$. On the other hand, when $\nu = 0$, i.e. in the absence of dissipation, one may show that any measure preserved by a.a. rotations e^{β^I} is invariant and that this property characterizes the invariant measures with finite characteristic functions [26]. In particular, the Gibbs measures $\sim \mathrm{e}^{-\mathrm{const.}\|T\|^2}\, d^dT$ are invariant under the flow (4.9) when $\nu = 0$. As we shall see, the situation is quite different for infinite number of degrees of freedom.

In the presence of white-noise sources and of dissipation, the 2-point equal time correlation function evolves according to the equation

$$\begin{aligned} < T(t)^{\otimes 2} > \,&=\, \mathrm{e}^{-(t-t_0)\,\mathcal{M}_2}\, T(t_0)^{\otimes 2} + \int_{t_0}^{t} ds\, \mathrm{e}^{-(t-s)\,\mathcal{M}_2}\, \mathcal{C} \\ &=\, \mathrm{e}^{-(t-t_0)\,\mathcal{M}_2}\, T(t_0)^{\otimes 2} + (1-\mathrm{e}^{-(t-t_0)\mathcal{M}_2})\,\mathcal{M}_2^{-1}\,\mathcal{C} \end{aligned} \tag{4.26}$$

which is a solution of the differential equation

$$\partial_t < T(t)^{\otimes 2} > = -\mathcal{M}_2 < T(t)^{\otimes 2} > + \,\mathcal{C}\;. \tag{4.27}$$

When $t_0 \to -\infty$, the term with $T(t_0)$ disappears due to the positivity of $\mathcal{M}_2$ (for $\nu > 0$) and we obtain

$$< T^{\otimes 2} > \,=\, \mathcal{M}_2^{-1}\mathcal{C}\;. \tag{4.28}$$

This is the time independent 2-point function in a steady state in which the quantities of the energy injected by the force and dissipated balance each other. Similarly, for the general equal time correlators in the presence of the external sources and of dissipation, the dependence on the initial conditions is wiped out when $t_0 \to -\infty$ due to the positivity of all $\mathcal{M}_N$. The higher stationary state equal-time correlation functions take the form:

$$\begin{aligned} &< T^{\otimes 2M+1} > \,=\, 0\;, \\ &< T^{\otimes 2M} > \;=\; \sum_{\sigma\,\in\,\mathcal{S}_{2M}\,/\,\mathcal{S}_M\times\mathcal{S}_2^M} \sigma\; F_{2M}\;, \end{aligned} \tag{4.29}$$

where $F_{2M} \in (\mathbf{R}^d)^{\otimes 2M}$ is a single 2-particle channel contribution. Permutations σ interchange the factors in $(\mathbf{R}^d)^{\otimes 2M}$, $\mathcal{S}_M$ is viewed as a set of permutations of the pairs $(2m-1, 2m)$, $m = 1, \ldots, M$, and $\mathcal{S}_2^M$ as that of permutations acting within the pairs so that the sum is over the pairings of $\{1, \ldots, 2M\}$ or 2-particle channels.

$$F_{2M} \;=\; \mathcal{M}_{2M}^{-1} \sum_{\sigma\in\mathcal{S}_M} \sigma\, \Big((\mathcal{M}_{2M-2}^{-1} \otimes 1_2)\; \cdots\; (\mathcal{M}_2^{-1} \otimes 1_{2M-2}) \Big)\, \mathcal{C}^{\otimes M}\;. \tag{4.30}$$

Exact expressions for non-equal times N-point correlators are also easy to obtain. They involve also the heat operators $\mathrm{e}^{-t\mathcal{M}_{N'}}$ with $N' < N$.

Let us discuss briefly some multibody combinatorics behind the solution (4.30). Let us denote by $\mathcal{P}_{2M}$ the set of pairs $(2m-1, 2m)$, $m = 1, \ldots, M$. We shall define the connected parts F^c_{2M} of the single channel correlator by the inductive formula

$$F_{2M} \;=\; \sum_{\substack{\text{partitions}\\ \Pi \text{ of } \mathcal{P}_{2M}}} \bigotimes_{\pi\in\Pi} F^c_\pi \tag{4.31}$$

in the, hopefully, self-explanatory notation. The connected $2M$-correlator $< T^{\otimes 2M} >^c$ is given by Eq. (4.29) with F^c_{2M} replacing F_{2M} on the right hand side. Note that, by virtue of the definition (4.21) of $\mathcal{M}_N$, for each partition Π of $\mathcal{P}_{2M}$, we may write

$$\mathcal{M}_{2M} \;=\; \sum_{\pi\in\Pi} \mathcal{M}_\pi - \sum_{\{\pi,\pi'\}\subset\Pi} \mathcal{L}_{\pi,\pi'} \tag{4.32}$$

where $\mathcal{M}_\pi$ is $\mathcal{M}_{2|\pi|}$ acting on the tensor factors of $(\mathbf{R}^d)^{\otimes 2M}$ labeled by the pairs of π and

$$\mathcal{L}_{\pi,\pi'} \;=\; \sum_{\substack{n\in\pi\\ n'\in\pi'}} (\mathcal{K})_{n,n'} \;. \tag{4.33}$$

The basic equation for the connected single 2-particle channel correlation functions F^c_{2M} is given by

Proposition 3. For $M > 1$,

$$\mathcal{M}_{2M}\, F^c_{2M} \;=\; \sum_{\Pi=\{\pi,\pi'\}} \mathcal{L}_{\pi,\pi'}\,(F^c_\pi \otimes F^c_{\pi'}) \;. \tag{4.34}$$

Note the connected character of the right hand side. Let us prove this relation. From the definition (4.30) of the single channel function,

$$\mathcal{M}_{2M}\, F_{2M} \;=\; \sum_{\substack{\Pi=\{\pi,\pi'\}\\ |\pi|=1}} \mathcal{C}_\pi \otimes F_{\pi'} \;. \tag{4.35}$$

On the other hand, using Eqs. (4.31) and (4.32), we obtain

$$\begin{aligned} \mathcal{M}_{2M}\, F^c_{2M} \;&=\; \mathcal{M}_{2M}\, F_{2M} - \sum_{\substack{\Pi\\ |\Pi|>1}} \mathcal{M}_{2M} \bigotimes_{\pi\in\Pi} F^c_\pi \\ &=\; \mathcal{M}_{2M}\, F_{2M} - \sum_{\substack{\Pi\\ |\Pi|>1}} \sum_{\pi\in\Pi} (\mathcal{M}_\pi\, F^c_\pi) \bigotimes \Big(\bigotimes_{\substack{\pi'\in\Pi\\ \pi'\neq\pi}} F^c_{\pi'} \Big) \\ &+\; \sum_{\substack{\Pi\\ |\Pi|>1}} \sum_{\{\pi,\pi'\}\subset\Pi} \big(\mathcal{L}_{\pi,\pi'}(F^c_\pi \otimes F^c_{\pi'})\big) \bigotimes \Big(\bigotimes_{\pi''\neq\pi,\pi'} F^c_{\pi''} \Big) \;. \end{aligned} \tag{4.36}$$

Notice that on the right hand side, in virtue of Eqs. (4.35) and (4.28), the 1$^{\rm st}$ term cancels with the contribution to the 2$^{\rm nd}$ one from π consisting of a single pair. Rewriting $\mathcal{M}_\pi\, F^c_\pi$ for the $|\pi| > 1$ contributions to the 2$^{\rm nd}$ term using Eq. (4.34) as the inductive hypothesis, we observe that all these contribution are cancelled by the 3$^{\rm rd}$ term. What is left from the 3$^{\rm rd}$ term after the cancellations gives exactly the right hand side of Eq. (4.34).

Although we worked directly with the evolution equation (4.9) in order to obtain expressions (4.28) and (4.30), one may obtain them as well using the MSR formalism which gives

$$< T^{\otimes 2M} >= \tfrac{1}{\mathcal{N}} \int T^{\otimes 2M}\, \mathrm{e}^{i\,(A,\,(\partial_t+\nu\,\alpha+\beta)T) - \frac{1}{2}\,(A,\mathcal{C}A) - \frac{1}{2}\,(\beta,\mathcal{B}^{-1}\beta)}\; DA\; DT\; D\beta \;. \tag{4.37}$$

Performing the Gaussian A- and T-integrals, we obtain

$$< T^{\otimes 2M} >= \frac{1}{N} \sum_{\substack{\text{parings} \\ \text{of } \{1,\dots,2M\}}} \int \prod_{\text{pairs}} (\partial_t + \nu\alpha + \beta)^{-1} \mathcal{C} (-\partial_t + \nu\alpha - \beta)^{-1} \, \mathrm{e}^{-\frac{1}{2}(\beta, \mathcal{B}^{-1}\beta)} \, D\beta \,. \quad (4.38)$$

If we expand

$$(\partial_t + \nu\alpha + \beta)^{-1} \;=\; \sum_{m=0}^{\infty} (-1)^m \, (\partial_t + \nu\alpha)^{-1} \left(\beta \, (\partial_t + \nu\alpha)^{-1} \right)^m \quad (4.39)$$

and use the fact that the kernel of $(\partial_t + \nu\alpha)^{-1}$ is equal to $\theta(t-t') \, \mathrm{e}^{-(t-t')\nu\alpha}$, the β-integral in (4.38) reduces to the previous calculation and it reproduces the expressions (4.28) for the 2-point function or (4.29) and (4.30) for the $2M$-correlator.

4.3 Eddy diffusion versus super-diffusion

The exact solution for the PS correlation functions obtained above in the case of finitely many degrees of freedom still makes sense for the infinite-dimensional case of the evolution equation (4.1). Instead of starting with this equation, and extending the previous analysis from the stochastic ODE case to the stochastic PDE one, which would require additional work, we shall directly study the infinite-dimensional version of the exact solution (4.29), (4.30) involving operators rather (generally unbounded) than finite matrices. The analysis of the expressions for the PS correlation functions obtained this way is more complicated than in finite dimensions, but also more interesting.

The covariance $\mathcal{B}$ of the Gaussian process $\beta = \mathbf{v} \cdot \nabla$ and the operator B related to the expectation of β^2, see Eqs. (4.16) and (4.17), are given by the kernels

$$\begin{aligned} \mathcal{B}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2) \;&=\; \mathcal{D}^{ij}(\mathbf{x}_1 - \mathbf{x}_2) \, \partial_i \delta(\mathbf{x}_1 - \mathbf{y}_1) \, \partial_j \delta(\mathbf{x}_2 - \mathbf{y}_2) \;, \\ B(\mathbf{x}, \mathbf{y}) \;&=\; -\tfrac{1}{2} \int \mathcal{B}(\mathbf{x}, \mathbf{z}; \mathbf{z}, \mathbf{y}) \, d^3\mathbf{z} \;=\; -\tfrac{1}{2} \mathcal{D}^{ij}(0) \, \partial_i \partial_j \, \delta(\mathbf{x} - \mathbf{y}) \;. \end{aligned} \quad (4.40)$$

Since $\mathcal{D}^{ij}(0) = \frac{1}{3} \delta^{ij} \, \mathcal{D}^{ll}(0)$, $B = -\frac{1}{6} \mathcal{D}^{ll}(0) \, \Delta$ and, consequently,

$$\mathcal{M}_1 \equiv \nu\alpha + B = -(\nu + \tfrac{1}{6}\mathcal{D}^{ll}(0)) \, \Delta \equiv -\nu_{\text{eff}} \, \Delta \,. \quad (4.41)$$

It follows from Eq. (4.23) with $N = 1$ that, in the absence of the sources, the expectation value of the scalar diffuses

$$\begin{aligned} < T(t, \mathbf{x}) > \;&=\; \int \mathrm{e}^{-(t-t_0)\mathcal{M}_1}(\mathbf{x}, \mathbf{y}) \, T(t_0, \mathbf{y}) \, d^3\mathbf{y} \\ &=\; \int \mathrm{e}^{(t-t_0)\nu_{\text{eff}}\Delta}(\mathbf{x}, \mathbf{y}) \, T(t_0, \mathbf{y}) \, d^3\mathbf{y} \end{aligned} \quad (4.42)$$

with the effective diffusion constant ν_{eff} composed of the molecular diffusivity ν and the **eddy diffusivity** $\frac{1}{6}\mathcal{D}^{ll}(0)$. For small m, the eddy diffusivity dominates and the diffusion is driven by the large distance scales (recall from Eq. (4.5) that $\mathcal{D}^{ll}(0) = \mathcal{O}(D_0 m^{-\kappa})$).

Operator $\mathcal{K}$ with kernel $\mathcal{K}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2) = \mathcal{B}(\mathbf{x}_1, \mathbf{y}_1; \mathbf{x}_2, \mathbf{y}_2)$ acts on functions g of six variables by

$$(\mathcal{K} g)(\mathbf{x}_1, \mathbf{x}_2) \;=\; \mathcal{D}^{ij}(\mathbf{x}_1 - \mathbf{x}_2) \, \partial_{x_1^i} \partial_{x_2^j} \, g(\mathbf{x}_1, \mathbf{x}_2) \;. \quad (4.43)$$

so that for the N-body operators $\mathcal{M}_N$ given by formula (4.21) we obtain

$$\begin{aligned}\mathcal{M}_N \;&=\; -\sum\limits_{n=1}^{N}(\nu\,\Delta_{\mathbf{x}_n}+\tfrac{1}{2}\mathcal{D}^{ij}(0)\,\partial_{x_n^i}\partial_{x_n^j})\;-\;\sum\limits_{n<n'}\mathcal{D}^{ij}(\mathbf{x}_n-\mathbf{x}_{n'})\,\partial_{x_n^i}\partial_{x_{n'}^j}\\ &=\;-\sum\limits_{n=1}^{N}\nu_{\rm eff}\,\Delta_{\mathbf{x}_n}\;-\;\sum\limits_{n<n'}\mathcal{D}^{ij}(\mathbf{x}_n-\mathbf{x}_{n'})\,\partial_{x_n^i}\partial_{x_{n'}^j}\;.\end{aligned}\tag{4.44}$$

In particular,

$$\mathcal{M}_2\;=\;-\nu(\Delta_1+\Delta_2)-\tfrac{1}{2}\mathcal{D}^{ij}(0)\,\partial_{x_1^i}\partial_{x_1^j}-\tfrac{1}{2}\mathcal{D}^{ij}(0)\,\partial_{x_2^i}\partial_{x_2^j}-\mathcal{D}^{ij}(\mathbf{x}_1-\mathbf{x}_2)\,\partial_{x_1^i}\partial_{x_2^j}\,.\tag{4.45}$$

Note that $\mathcal{M}_2$ commutes with (three-dimensional) translations and in the action on translation-invariant functions of $\mathbf{x}_1-\mathbf{x}_2\equiv\mathbf{x}$ reduces to

$$\mathcal{M}_2\;=\;-2\nu\,\Delta\;-\;\widetilde{\mathcal{D}}^{ij}(\mathbf{x})\,\partial_i\partial_j\;.\tag{4.46}$$

Since $\widetilde{\mathcal{D}}^{ij}(\mathbf{x})\equiv\mathcal{D}^{ij}(0)-\mathcal{D}^{ij}(\mathbf{x})$ has an $m\to 0$ limit, so does the operator $\mathcal{M}_2$ in the action on translation-invariant functions and when $\nu\to 0$, it becomes a singular elliptic operator

$$\mathcal{M}_2^{\rm sc}\;\;=\;\;-D_1\left((2+\kappa)\,\delta^{ij}\,|x|^\kappa\;-\;\kappa\,x^i\,x^j\,|x|^{\kappa-2}\right)\partial_i\partial_j\;,\tag{4.47}$$

where $D_1\;\equiv\;\frac{\Gamma((2-\kappa)/2)}{2^{2+\kappa}\pi^{3/2}\kappa(3+\kappa)}\,D_0$, see Eq. (4.6). The dimension of D_1, as that of D_0, is equal to $\frac{length^{2-\kappa}}{time}$.

Similarly for $N>2$, in the action on the translationally invariant sector,

$$\mathcal{M}_N\;=\;-\nu\sum\limits_{n=1}^{N}\Delta_{\mathbf{x}_n}\;+\;\sum\limits_{n<n'}\widetilde{\mathcal{D}}^{ij}(\mathbf{x}_n-\mathbf{x}_{n'})\,\partial_{x_n^i}\partial_{x_{n'}^j}\tag{4.48}$$

and has an $m\to 0$ and $\nu\to 0$ limit which is a singular elliptic operator $\mathcal{M}_N^{\rm sc}$. This has important consequences. Although the time evolution given by Eq. (4.23) of fixed initial conditions is dominated for small m by the eddy diffusivity diverging as $m\to 0$, the time evolution governed by Eq. (4.25) for random initial data with a translationally invariant distribution behaves well in the limit when $m\to 0$ and $\nu\to 0$. In particular, it follows from the scaling properties of operators $\mathcal{M}_N^{\rm sc}$ that, if the translationally invariant initial moments $<\prod_n T(t_0,\mathbf{x}_n)>$ are fast decaying in the difference variables (weakly correlated initial data) then, for $m=0$, the rescaled correlations

$$\lambda^{\frac{3N}{2(2-\kappa)}}\;<\prod\limits_{n=1}^{N}T(\lambda t,\,\lambda^{\frac{1}{2-\kappa}}\mathbf{x}_n)>\tag{4.49}$$

have a non-trivial limit when $\lambda\to\infty$. In other words, the equal-time correlators decay **super-diffusively** in the absence of sources, like $t^{-\frac{3N}{2(2-\kappa)}}$, i.e. faster than diffusively (like $t^{-\frac{3N}{4}}$). Similar behavior in a simplified model of passive advection was discussed in [13].

Let us look closer at the operator $\mathcal{M}_2$ in the translationally invariant sector. Since the limit $m\to 0$ presents no difficulty in that case, we may directly work in the $m=0$ case, i.e. with operator $\mathcal{M}_2=-2\nu\,\Delta+\mathcal{M}_2^{\rm sc}$, see Eq. (4.47). In the radial variables

$$\mathcal{M}_2\;=\;-\tfrac{2}{r^2}\,\partial_r\,(\nu\,r^2\,+\,D_1\,r^{2+\kappa})\,\partial_r\,+\,\tfrac{l(l+1)}{r^2}\,(2\nu\,+\,D_1(2+\kappa)\,r^{\kappa-2})\;,\tag{4.50}$$

where l is the angular momentum ($\mathcal{M}_2$ commutes with rotations). As for the resolvents $(i\omega + \mathcal{M}_2)^{-1}$, they may be explicitly computed in the translationally and rotationally invariant ($l = 0$) sector when $\nu \to 0$. In particular, one obtains for the kernel

$$(i\omega + \mathcal{M}_2^{\rm sc})^{-1}(\mathbf{x}, 0) \;=\; \omega^{\frac{1+\kappa}{2-\kappa}}\, f(\omega^{\frac{1}{2}}|\mathbf{x}|^{\frac{2-\kappa}{2}}) \tag{4.51}$$

with an exponentially decaying function f, $f(y) \sim y^{-\frac{2(1+\kappa)}{2-\kappa}}$ for small y. f may be expressed in terms of the Hankel functions. This result has an important implication. Recall that in finite dimensions the invariant measures for the stochastic evolution of the initial data were characterized by the symmetry generated by rotations e^{β^I}. Suppose that, as for a finite number of degrees of freedom, the evolution (4.25) preserves in the limit $\nu \to 0$ the white noise Gibbs measure $\sim\ \mathrm{e}^{-{\rm const.}\|T\|^2}\, DT$ with the 2-point function

$$< T(\mathbf{x})\, T(\mathbf{y}) > \sim \delta(\mathbf{x} - \mathbf{y})\ . \tag{4.52}$$

This would imply that

$$\int g(\mathbf{x})\, (i\omega + \mathcal{M}_2^{\rm sc})^{-1}(\mathbf{x}, 0)\, d^3\mathbf{x} \;=\; \frac{1}{i\omega}\, g(0) \tag{4.53}$$

for any test function g, which is in contradiction with Eq. (4.51). Taking Fourier transforms in space and time of the latter, one obtains

$$\int \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\ \mathrm{e}^{-t\,\mathcal{M}_2^{\rm sc}}(\mathbf{x}, 0)\, d^3\mathbf{x} \;=\; \rho(t^{\frac{1}{2-\kappa}}\,|\mathbf{k}|)\ , \tag{4.54}$$

with $\rho(0) = 1$ and ρ decaying at infinity, instead of a constant as would be implied by Eq. (4.53). One should compare the relation (4.54) to the relation

$$\int \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\ \mathrm{e}^{-t\Delta}(\mathbf{x}, 0)\, d^3\mathbf{x} \;=\; \mathrm{e}^{-t|\mathbf{k}|^2} \tag{4.55}$$

describing the diffusive decay of the energy spectrum. For example, Eq. (4.54) implies that for homogeneous, fast decaying initial data 2-point function,

$$\int < T(t, 0)\, T(t, \mathbf{x}) >\ \mathbf{x}^2\, d^3\mathbf{x} \quad \sim \quad t^{\frac{2}{2-\kappa}} \tag{4.56}$$

instead of being proportional to t: again a super-diffusive behavior. The reason of the failure of the Gibbs measure to be invariant under the evolution (4.25) for infinitely many degrees of freedom is the energy cascade towards degrees of freedom with smaller and smaller wavenumbers, see below. This failure shows that, for the infinite number of degrees of freedom, the symmetry group generated by the exponentials e^{β^I}, which for $\beta = \mathbf{v}\cdot\nabla$ is the group of volume preserving diffeomorphisms of $\mathbf{R}^3$, is broken. It is not excluded, however, that this symmetry may still be useful in analysis of the model.

4.4 Quasi-Lagrangian approach to passive scalar

PS constitutes a perfect model to test the quasi-Lagrangian ideas mentioned at the end of Section 4. The scalar density $T'(t, \mathbf{x}) \equiv T(t, \mathbf{x} + \mathbf{x}(t))$ viewed from a point moving in the fluid satisfies the equation

$$\partial_t T' \,+\, (\mathbf{v}' - \mathbf{v}_0')T' \,-\, \nu\Delta T' \;=\; f' \tag{4.57}$$

where $\mathbf{v}'(t,\mathbf{x}) \equiv \mathbf{v}(t,\mathbf{x}+\mathbf{x}(t))$ and $f'(t,\mathbf{x}) \equiv f(t,\mathbf{x}+\mathbf{x}(t))$ in the notation of Section 4. It is again a stochastic equation of the type (4.9) with $\beta = (\mathbf{v}' - \mathbf{v}'_0)\cdot\nabla$. One may show that for Gaussian homogeneous $\mathbf{v}(t,\mathbf{x})$ and $f(t,\mathbf{x})$, white noise in time, the distributions of the random fields $\mathbf{v}'(t,\mathbf{x})$ and $f'(t,\mathbf{x})$ obtained by the quasi-Lagrangian transformation become the same as those of $\mathbf{v}(t,\mathbf{x})$ and $f(t,\mathbf{x})$ asymptotically for large times. For $\mathbf{v}'(t,\mathbf{x})$ and $\mathbf{f}'(t,\mathbf{x})$ distributed as $\mathbf{v}(t,\mathbf{x})$ and $f(t,\mathbf{x})$ before, we may produce the exact solution for the correlation functions of T' by replacing the operators $\mathcal{M}_N$ by new operators $\mathcal{M}'_N$ obtained by changing the velocity correlation function (4.2) to

$$
\begin{aligned}
&< (v'^i(t,\mathbf{x}) - v'^i(t,0))\,(v'^j(t',\mathbf{x}') - v'^j(t',0)) > \\
&\qquad = \delta(t-t')\,\Big(D^{ij}(\mathbf{x}-\mathbf{x}') - D^{ij}(\mathbf{x}) - D^{ij}(\mathbf{x}') + D^{ij}(0)\Big) \\
&\qquad = -\,\delta(t-t')\,\Big(\widetilde{D}^{ij}(\mathbf{x}-\mathbf{x}') - \widetilde{D}^{ij}(\mathbf{x}) - \widetilde{D}^{ij}(\mathbf{x}')\Big)\ .
\end{aligned}
\tag{4.58}
$$

Notice that only the subtracted functions $\widetilde{D}^{ij}$ with the regular $m \to 0$ limit enter the correlator (4.58). One obtains

$$
\begin{aligned}
\mathcal{M}'_N \;=\; &\nu \sum_{n=1}^{N} \Delta_{\mathbf{x}_n} - \sum_{n=1}^{N} \widetilde{D}^{ij}(\mathbf{x}_n)\,\partial_{x_n^i}\,\partial_{x_n^j} \\
&+ \sum_{n<n'} \Big(\widetilde{D}^{ij}(\mathbf{x}_n - \mathbf{x}_{n'}) - \widetilde{D}^{ij}(\mathbf{x}_n) - \widetilde{D}^{ij}(\mathbf{x}_{n'})\Big)\,\partial_{x_n^i}\,\partial_{x_{n'}^j}\ .
\end{aligned}
\tag{4.59}
$$

The crucial observation is that in the action on translationally invariant functions operators $\mathcal{M}'_N$ and $\mathcal{M}_N$, the latter given by Eq. (4.48), coincide (the terms with $\widetilde{D}^{ij}(\mathbf{x}_n)$ cancel). Hence for the PS, the stationary equal time quasi-Lagrangian correlators will be equal to the original (Eulerian) ones for the homogeneous sources. This is not the case for the non-equal time correlators which involve the heat operators $\mathrm{e}^{-t\mathcal{M}_N}$ ($\mathrm{e}^{-t\mathcal{M}'_N}$) in the non-translationally invariant sector. The latter are governed by the eddy diffusivity, i.e. by the integral scale in the Eulerian case but show a universal behavior in the inertial range in the quasi-Lagrangian description. The quasi-Lagrangian picture is also better behaved from the point of view of the standard perturbative approach which consists of treating the 2-particle terms in $\mathcal{M}_N$ as a perturbation of the 1-particle ones, see Eq. (4.21). In the Eulerian approach, the unperturbed terms are dominated by the diverging eddy diffusivity whereas in the quasi-Lagrangian one the 1-particle terms have a regular limit when $m \to 0$, compare Eqs. (4.44) and (4.59). Thus the quasi-Lagrangian perturbation expansion seems to be a better tool for the analysis of the behavior of the model.

5 Steady state of the passive scalar

5.1 Universality of the 2-point function

Let us pass to the study of the stationary state 2-point function $< T(\mathbf{x}_1)\,T(\mathbf{x}_2) > \equiv F_2(|\mathbf{x}_1 - \mathbf{x}_2|)$ in the presence of the random sources with a rotationally invariant covariance $\mathcal{C}_L = \mathcal{C}(\cdot/L)$, see Eq. (4.7). F_2 satisfies the differential equation

$$
(\mathcal{M}_2\,F_2)(r) \;=\; -\tfrac{2}{r^2}\,\partial_r\,(\nu\,r^2 + D_1\,r^{2+\kappa})\,\partial_r\,F_2(r) \;=\; \mathcal{C}_L(r)
\tag{5.1}
$$

which is implied by Eq. (4.28). Integrating once, we obtain

$$\partial_r F_2(r) \;=\; -\tfrac{1}{2}\,\frac{\int\limits_0^r \mathcal{C}(\rho/L)\,\rho^2\,d\rho}{\nu\, r^2 \,+\, D_1\, r^{2+\kappa}}\;, \tag{5.2}$$

where the constant of integration has been chosen so as to obtain $\partial_r F_2$ vanishing at zero as required if F_2 is to be a function on $\mathbf{R}^3$ regular at zero. One more integral gives

$$F_2(r) \;=\; \tfrac{1}{2}\int\limits_r^\infty \frac{\int\limits_0^\rho \mathcal{C}(\rho'/L)\,\rho'^2\,d\rho'}{\nu\,\rho^2 \,+\, D_1\,\rho^{2+\kappa}}\;d\rho\;, \tag{5.3}$$

where the choice of integration constant assures that F_2 decays at infinity. We want to study the behavior of $F_2(r)$ when the integral scale $L\to\infty$. To this end, let us rewrite the ρ-integral from r to ∞ as a difference of the integral from 0 to ∞ and from 0 to r:

$$F_2(r) \;=\; \tfrac{1}{2}\,L^{2-\kappa}\int\limits_0^\infty \frac{\int\limits_0^\rho \mathcal{C}(\rho')\,\rho'^2\,d\rho'}{L^{-2-\kappa}\,\nu\,\rho^2 \,+\, D_1\,\rho^{2+\kappa}}\;d\rho \;-\; \tfrac{1}{2}\int\limits_0^r \frac{\int\limits_0^\rho \mathcal{C}(\rho'/L)\,\rho'^2\,d\rho'}{\nu\,\rho^2 \,+\, D_1\,\rho^{2+\kappa}}\;d\rho\;, \tag{5.4}$$

The first of the integrals is a $\mathcal{C}$-dependent constant diverging when $L\to\infty$. On the other hand, the integral from 0 to r has a limit when $L\to\infty$ equal to

$$-\,\tfrac{1}{6}\,\mathcal{C}(0)\int\limits_0^r \frac{\rho^3\,d\rho}{\nu\rho^2 + D_1\rho^{2+\kappa}} \tag{5.5}$$

which is universal (i.e. depends only on $\mathcal{C}(0)$ which, as we shall see in the next Section, equals twice the energy dissipation rate ϵ of the scalar). Note that the non-universal term (a constant) is annihilated by $\mathcal{M}_2$. This has to be so if Eq. (5.1) is to be satisfied. The right hand side becomes universal in the limit $L\to\infty$ so all non-universal terms in $F_2(r)$ surviving in this limit have to be annihilated by $\mathcal{M}_2$. We shall see this general mechanism limiting possible non-universal terms also for the higher point functions. The constant term of F_2 is automatically subtracted in the $2^{\rm nd}$ structure function:

$$S_2(r) \,=\, 2(F_2(0) - F_2(r)) \;=\; \tfrac{2}{3}\,\epsilon\int\limits_0^r \frac{\rho^3\,d\rho}{\nu\rho^2 + D_1\rho^{2+\kappa}} \tag{5.6}$$

where the last equality holds for infinite L. In the $\nu\to 0$ limit,

$$S_2(r) \;=\; \frac{2\,\epsilon}{3(2-\kappa)\,D_1}\;r^{2-\kappa} \tag{5.7}$$

and the same universal result holds approximately in the whole inertial range $\eta \ll r \ll L$ where at the Kolmogorov scale $\eta = \mathcal{O}((\nu/D_1)^{1/\kappa})$ the term $\nu\, r^2$ becomes comparable to $D_1\,\rho^{2+\kappa}$. The $2^{\rm nd}$ structure function exponent $\zeta_2 = 2-\kappa$.

It is instructive to repeat the same analysis for anisotropic sources, i.e. without assuming that the correlation $\mathcal{C}_L$ is rotationally invariant. We shall have then to solve the equation

$$\mathcal{M}_2\,F_2 \,=\, \mathcal{C}_L \tag{5.8}$$

in other angular momentum sectors. This is an easy exercise [26] and one still obtains, for m and ν equal to zero, the result

$$F_2(\mathbf{x}) \;-\; \gamma\, L^{2-\kappa} \;\underset{L\to\infty}{\longrightarrow}\; -\;\tfrac{\epsilon}{3(2-\kappa)\,D_1}\,|\mathbf{x}|^{2-\kappa} \;\equiv\; F_2^{\rm sc}(\mathbf{x})\;, \tag{5.9}$$

where γ is a $\mathcal{C}$-dependent constant. In the case of a complex PS, however, further subtractions are needed when $0<\kappa<1$. For $F_2(\mathbf{x}) \equiv < T(0)\,\overline{T(\mathbf{x})} >$, one shows then [26] that

$$F_2(\mathbf{x}) \;-\; \gamma\, L^{2-\kappa} \;-\; \gamma_i\, x^i\, L^{1-\kappa} \;\underset{L\to\infty}{\longrightarrow}\; -\;\tfrac{\epsilon}{3(2-\kappa)\,D_1}\,|\mathbf{x}|^{2-\kappa}\;, \tag{5.10}$$

where real γ and imaginary γ_i are $\mathcal{C}$-dependent constants. The γ_i-terms are present only if $\mathcal{C}$ has angular momentum 1 components and this is impossible in the real case where only even l's appear[4]. Again, the non-universal terms are homogeneous zero modes of $\mathcal{M}_2$ which annihilates constant and linear functions, in accordance with the general argument. Similarly as the constant, the linear terms appearing with the coefficient $L^{1-\kappa}$ in the presence of anisotropic sources for the complex scalar drop out from the $2^{\rm nd}$ structure function $< |T(\mathbf{x})-T(0)|^2 >= 2\,F_2(0)-F_2(\mathbf{x})-F_2(-\mathbf{x})$ and do not spoil the inertial range universality of the latter.

5.2 Energy cascade

Let us look at the energy balance of the scalar. The dynamical equation (4.27) takes the form

$$\partial_t F_2(r) \;=\; -(\mathcal{M}_2\, F_2)(r) + \mathcal{C}_L(r)\;. \tag{5.11}$$

and reduces in the stationary state to Eq. (5.1). In particular, the mean energy density $e \equiv < \frac{1}{2}T^2 >$ satisfies

$$\begin{aligned}\partial_t e \;&=\; \tfrac{1}{2}\,\partial_t F_2(0) \;=\; -\tfrac{1}{2}\,(\mathcal{M}_2\, F_2)(0) \,+\, \tfrac{1}{2}\,\mathcal{C}(0)\\ &=\; \nu(\Delta F_2)(0) \,+\, \tfrac{1}{2}\,\mathcal{C}(0)\end{aligned} \tag{5.12}$$

(only the $-2\nu\Delta$ term of $\mathcal{M}_2$ contributes at $r=0$).

$$\epsilon \;\equiv\; -\nu(\Delta F_2)(0) \;=\; -\nu\tfrac{1}{r^2}\,\partial_r r^2\,\partial_r F_2(r)|_{r=0} \tag{5.13}$$

is the mean dissipation rate and $\frac{1}{2}\mathcal{C}(0)$ is the mean injection rate of energy (both intensive). In the steady state the energy balance reduces to the equality of the rates :

$$\epsilon \;=\; \tfrac{1}{2}\,\mathcal{C}(0)\;. \tag{5.14}$$

It is, of course, satisfied by our solution for the steady state 2-point function.

We would like to see how the energy is distributed among different wavenumbers following the schematic discussion of Section 1.2. For the mean energy density of modes with $|\mathbf{k}|\leq K$, $e_{\leq K} \equiv < \frac{1}{2}\,T_{\leq K}^2 >$, we obtain

$$e_{\leq K} \;=\; \tfrac{1}{2}\int\limits_{|\mathbf{k}|\leq K} \Big(\int \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{y}}\; F_2(\mathbf{y})\; d^3\mathbf{y}\Big)\; \tfrac{d^3\mathbf{k}}{(2\pi)^3} \;\equiv\; \int\limits_0^K e(k)\,dk \tag{5.15}$$

[4]K.G. would like to thank R. Kraichnan for pointing this out to him

with the energy spectrum

$$e(k) \sim k^{-3+\kappa} \tag{5.16}$$

in the steady state inertial range. The dynamical equation (5.11) implies that

$$\partial_t e_{\leq K} = -\epsilon_{\leq K} + \varphi_{\leq K} - \pi_K , \tag{5.17}$$

where

$$\begin{aligned}\epsilon_{\leq K} &\equiv -\nu \int\limits_{|\mathbf{k}|\leq K} \left(\int \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{y}}\, \Delta F_2(\mathbf{y})\, d^3\mathbf{y}\right) \tfrac{d^3\mathbf{k}}{(2\pi)^3} \\ &= -\tfrac{2}{\pi}\nu \int\limits_0^\infty \tfrac{\sin(Kr) - Kr\,\cos(Kr)}{r^3}\ \partial_r r^2 \partial_r F_2(r)\, dr\end{aligned} \tag{5.18}$$

is the mean energy dissipation rate in the modes with $|\mathbf{k}| \leq K$ and

$$\varphi_{\leq K} \equiv \tfrac{1}{2} \int\limits_{|\mathbf{k}|\leq K} \left(\int \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{y}}\, \mathcal{C}_L(\mathbf{y})\, d^3\mathbf{y}\right) \tfrac{d^3\mathbf{k}}{(2\pi)^3} = \tfrac{1}{\pi} \int\limits_0^\infty \tfrac{\sin(Kr) - Kr\,\cos(Kr)}{r}\ \mathcal{C}(r/L)\, dr \tag{5.19}$$

is the mean energy injection rate into $|\mathbf{k}| \leq K$.

$$\pi_K \equiv -\tfrac{2D_1}{\pi} \int\limits_0^\infty \tfrac{\sin(Kr) - Kr\,\cos(Kr)}{r^3}\ \partial_r r^{2+\kappa} \partial_r F_2(r)\, dr \tag{5.20}$$

comes from the contributions of $(\mathcal{M}_2^{\rm sc} F_2)(r)$ to (5.11). In view of the relation (5.17), it is natural to interpret π_K as the mean energy flux out of the modes with $|\mathbf{k}| \leq K$, compare Eq. (2.11) in Sect. 2. In particular, in the forced steady state, we obtain the energy balance equation

$$\varphi_{\leq K} = \epsilon_{\leq K} + \pi_K , \tag{5.21}$$

compare Eq. (2.12).

With the Fourier transform of $\mathcal{C}(\cdot/L)$ concentrated around the zero wavenumbers, the energy injection is essentially limited to small wavenumbers and the mean injection rate $\varphi_{\leq K}$ is approximately constant for $K \gg L^{-1}$ and equal to its $K = \infty$ value $\frac{1}{2}\mathcal{C}(0)$. On the other hand, the mean dissipation rate $\epsilon_{\leq K}$ may be easily showed for $\kappa > 1$ to satisfy

$$\epsilon_{\leq K} \leq \mathcal{O}(\frac{\nu\epsilon}{D_1} K^\kappa) . \tag{5.22}$$

Indeed, it follows from Eq. (5.2) that $|\partial_r r^2 \partial_r F_2(r)| \leq \text{const.}\ D_1^{-1} r^{2-\kappa}$. Estimating

$$|\sin(Kr) - Kr\cos(Kr)| \leq \begin{cases} \text{const.}\ (Kr)^3 & \text{for} \quad r \leq K^{-1} , \\ \text{const.}\ Kr & \text{for} \quad r \geq K^{-1} , \end{cases} \tag{5.23}$$

we obtain

$$\epsilon_{\leq K} \leq \text{const.} \frac{\nu\epsilon}{D_1} \left(K^3 \int\limits_0^{1/K} r^{2-\kappa} dr + K \int\limits_{1/K}^\infty r^{-\kappa} dr \right) \tag{5.24}$$

which implies the desired bound (5.22) for $\kappa > 1$. For $\kappa \leq 1$, a better estimate for $|\partial_r r^2 \partial_r F_2(r)|$ for large r using the decay of $\mathcal{C}$ implies a somewhat worse bound

$$\epsilon_{\leq K} \leq \mathcal{O}(\frac{\nu\epsilon}{D_1} K^\kappa) + \mathcal{O}(\frac{\nu\epsilon\, L^{1-\kappa}}{D_1} K) . \tag{5.25}$$

As a consequence, for $\kappa > 1$, the mean dissipation rate $\epsilon_{\leq K}$ stays very small relative to ϵ as long as $K \ll \eta^{-1}$ where $\eta = \mathcal{O}((\nu/D_1)^{1/\kappa})$ is the Kolmogorov scale. For $\kappa \leq 1$ this holds at least for $K \ll \nu^{-1} D_1 L^{\kappa-1}$. The energy balance equation (5.21), implies then that, for $\kappa > 1$, $L^{-1} \ll K \ll \eta^{-1}$, the mean energy flux π_K is almost equal to the mean injection rate $\varphi_{\leq K}$ and hence almost constant and equal to ϵ in the inertial range. For $\kappa \geq 1$ the same is true at least for $L^{-1} \ll K \ll \nu^{-1} D_1 L^{\kappa-1}$. This confirms an intuitive picture of the non-dissipative energy cascade through the decreasing distance scales from the integral scale L where the energy is injected to the Kolmogorov scale η at least for $\kappa > 1$. At the end the energy is dissipated on distances shorter than η. What happens for $\kappa \leq 1$ and K between $\mathcal{O}(\nu^{-1} D_1 L^{\kappa-1})$ and $\mathcal{O}(\eta^{-1})$ remains to be understood.

5.3 Gaussian case

It will be instructive to look at the case when $\kappa = 0$ with

$$\widetilde{D}^{ij} = 2\, D_1\, \delta^{ij}\ . \tag{5.26}$$

(Note that having finite D_1 requires infinitesimal D_0 when $\kappa \to 0$ in order to renormalize the ultraviolet divergence in the momentum-space integral (4.3); D_0 will never show up below.) Although quite trivial, the case $\kappa = 0$ appears to be instructive and will serve below as the departure point for an expansion in powers of κ. We immediately obtain for the $\kappa = 0$ operators:

$$\mathcal{M}_2^{\rm sc} = 2\, D_1 \nabla_{\mathbf{x}_1} \cdot \nabla_{\mathbf{x}_2} = -2\, D_1 \Delta_{\mathbf{X}}\ , \tag{5.27}$$

$$\begin{aligned}\mathcal{M}_4^{\rm sc} &= 2\, D_1 \sum_{1\leq n<n'\leq 4} \nabla_{\mathbf{x}_n} \cdot \nabla_{\mathbf{x}_{n'}} \\ &= -2\, D_1 \left(\Delta_{\mathbf{X}} + \Delta_{\mathbf{Y}} + \Delta_{\mathbf{Z}} - \nabla_{\mathbf{X}} \cdot \nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}} \cdot \nabla_{\mathbf{Z}}\right)\end{aligned} \tag{5.28}$$

in the difference variables $\mathbf{X} \equiv \mathbf{x}_1 - \mathbf{x}_2$, $\mathbf{Y} \equiv \mathbf{x}_2 - \mathbf{x}_3$, $\mathbf{Z} = \mathbf{x}_3 - \mathbf{x}_4$. The 2-point function

$$F_2(\mathbf{x}) = ({\mathcal{M}_2^{\rm sc}}^{-1} \mathcal{C}_L)(\mathbf{x}) = \tfrac{1}{2D_1} \int \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\, |\mathbf{k}|^{-2}\, \widehat{\mathcal{C}}_L(\mathbf{k})\, \tfrac{d^3\mathbf{k}}{(2\pi)^3}\ , \tag{5.29}$$

where

$$\widehat{\mathcal{C}}_L(\mathbf{k}) = \int \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\, \mathcal{C}(\mathbf{x}/L)\, d^3\mathbf{x} = L^3\, \widehat{\mathcal{C}}(L\mathbf{k})\ . \tag{5.30}$$

Hence

$$F_2(\mathbf{x}) = \tfrac{1}{2D_1}\, L^2 \int \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}/L}\, |\mathbf{k}|^{-2}\, \widehat{\mathcal{C}}(\mathbf{k})\, \tfrac{d^3\mathbf{k}}{(2\pi)^3}\ . \tag{5.31}$$

The large L behavior may be obtained by expanding $\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}/L}$ to the second order:

$$\begin{aligned}F_2(\mathbf{x}) &= \tfrac{1}{2D_1}\, L^2 \int |\mathbf{k}|^{-2}\, \widehat{\mathcal{C}}(\mathbf{k})\, \tfrac{d^3\mathbf{k}}{(2\pi)^3} + \tfrac{1}{2D_1}\, L x^j \int i k_j |\mathbf{k}|^{-2}\, \widehat{\mathcal{C}}(\mathbf{k})\, \tfrac{d^3\mathbf{k}}{(2\pi)^3} \\ &- \tfrac{1}{4D_1}\, x^i x^j \int k_i k_j |\mathbf{k}|^{-2}\, \widehat{\mathcal{C}}(k)\, \tfrac{d^3\mathbf{k}}{(2\pi)^3} + \mathcal{O}(L^{-1}) \\ &= \tfrac{1}{2D_1}\, ((-\Delta)^{-1}\mathcal{C})(0) + \tfrac{1}{2D_1}\, (\partial_i(-\Delta)^{-1}\mathcal{C})(0)\, x^i \\ &+ \tfrac{1}{4D_1}\, ((\partial_i\partial_j(-\Delta)^{-1} - \tfrac{1}{3}\delta_{ij})\, \mathcal{C})(0)\, x^i x^j + \tfrac{1}{12 D_1}\, \mathcal{C}(0)\, |x|^2 + \mathcal{O}(L^{-1})\ .\end{aligned} \tag{5.32}$$

Thus, up to a constant and linear terms (the latter absent for real $\mathcal{C}$) with $\mathcal{C}$-dependent, L-diverging coefficients and a quadratic zero mode of $\mathcal{M}_2^{\rm sc}$ with a $\mathcal{C}$-dependent coefficient, we obtain in the limit $L \to \infty$ the universal scaling solution $F_2^{\rm sc}(\mathbf{x}) = \frac{\epsilon}{6D_1}|\mathbf{x}|^2$, in accordance with the general result (5.9). Note that the separation of the scaling solution from the zero mode of the same homogeneity is ambiguous.

The 4-point function in the stationary state is given by Eqs. (4.29) and (4.30) for $M=2$, i.e. it may be written as a sum of contributions of three 2-particle channels:

$$< \prod_{n=1}^{4} T(\mathbf{x}_n) >= F_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) + F_4(\mathbf{x}_1,\mathbf{x}_3,\mathbf{x}_2,\mathbf{x}_4) + F_4(\mathbf{x}_1,\mathbf{x}_4,\mathbf{x}_2,\mathbf{x}_3) \tag{5.33}$$

with the single channel function

$$F_4 = \mathcal{M}_4^{-1}\,(\mathcal{M}_2^{-1}\otimes 1 + 1\otimes\mathcal{M}_2^{-1})\,\mathcal{C}_L\otimes\mathcal{C}_L\ . \tag{5.34}$$

Working in the difference variables, we obtain for $\kappa = 0$ using the scaling operators (5.27), (5.28):

$$\begin{aligned} F_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) &= \tfrac{1}{4D_1^2}\int \mathrm{e}^{i(\mathbf{k}_1\cdot\mathbf{x}_1+\ldots+\mathbf{k}_4\cdot\mathbf{x}_4)}\,(\sum_{n<n'}\mathbf{k}_n\cdot\mathbf{k}_{n'})^{-1} \\ &\quad\cdot((\mathbf{k}_1\cdot\mathbf{k}_2)^{-1} + (\mathbf{k}_3\cdot\mathbf{k}_4))^{-1}\,\widehat{\mathcal{C}}_L(\mathbf{k}_1)\,\widehat{\mathcal{C}}_L(\mathbf{k}_3) \\ &\quad\cdot(2\pi)^3\,\delta(\mathbf{k}_1+\mathbf{k}_2)\,(2\pi)^3\,\delta(\mathbf{k}_3+\mathbf{k}_4)\,\prod_{n=1}^{4}\tfrac{d^3\mathbf{k}_n}{(2\pi)^3} \\ &= \tfrac{1}{4D_1^2}\int \mathrm{e}^{i(\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)+\mathbf{k}_3\cdot(\mathbf{x}_3-\mathbf{x}_4))}\,(\mathbf{k}_1^2+\mathbf{k}_3^2)^{-1}\,((\mathbf{k}_1^2)^{-1}+(\mathbf{k}_3^2)^{-1}) \\ &\qquad\cdot\widehat{\mathcal{C}}_L(\mathbf{k}_1)\,\widehat{\mathcal{C}}_L(\mathbf{k}_3)\,\tfrac{d^3\mathbf{k}_1}{(2\pi)^3}\,\tfrac{d^3\mathbf{k}_3}{(2\pi)^3} \\ &= F_2(\mathbf{x}_1-\mathbf{x}_2)\,F_2(\mathbf{x}_3-\mathbf{x}_4)\ , \end{aligned} \tag{5.35}$$

see Eq. (5.29). It follows from Eq. (5.33) that for $\kappa = 0$ the equal time 4-point function is expressed by the 2-point function by the standard Gaussian formula. This remains true for the higher point functions. Let us consider the 6-point function

$$< \prod_{n=1}^{6} T(\mathbf{x}_n) > = \sum_{\sigma\in\mathcal{S}_6/\mathcal{S}_3\times\mathcal{S}_2^3} F_6(\mathbf{x}_{\sigma(1)},\ldots,\mathbf{x}_{\sigma(6)})\ , \tag{5.36}$$

where F_6 is the contribution of a single 2-particle channel,

$$F_6 = \mathcal{M}_6^{-1}\sum_{\sigma\in\mathcal{S}_3}\sigma\left((\mathcal{M}_4^{-1}\otimes 1_2)\,(\mathcal{M}_2^{-1}\otimes 1_4)\right)\mathcal{C}_L\otimes\mathcal{C}_L\ , \tag{5.37}$$

Explicitly, for $\kappa = 0$,

$$\begin{aligned} F_6(\mathbf{x}_1,\ldots,\mathbf{x}_6) &= \tfrac{1}{8D_1^3}\int \mathrm{e}^{i(\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)+\mathbf{k}_3\cdot(\mathbf{x}_3-\mathbf{x}_4)+\mathbf{k}_5\cdot(\mathbf{x}_5-\mathbf{x}_6))}\,(\mathbf{k}_1^2+\mathbf{k}_3^2+\mathbf{k}_5^2)^{-1} \\ &\quad\cdot((\mathbf{k}_1^2+\mathbf{k}_3^2)^{-1}+(\mathbf{k}_1^2+\mathbf{k}_5^2)^{-1}+(\mathbf{k}_3^2+\mathbf{k}_5^2)^{-1})\,((\mathbf{k}_1^2)^{-1} \\ &\quad+(\mathbf{k}_3^2)^{-1}+(\mathbf{k}_5^2)^{-1})\,\widehat{\mathcal{C}}_L(\mathbf{k}_1)\,\widehat{\mathcal{C}}_L(\mathbf{k}_3)\,\widehat{\mathcal{C}}_L(\mathbf{k}_5)\,\tfrac{d^3\mathbf{k}_1}{(2\pi)^3}\,\tfrac{d^3\mathbf{k}_3}{(2\pi)^3}\,\tfrac{d^3\mathbf{k}_5}{(2\pi)^3} \\ &= F_2(\mathbf{x}_1-\mathbf{x}_2)\,F_2(\mathbf{x}_3-\mathbf{x}_4)\,F_2(\mathbf{x}_5-\mathbf{x}_6)\ . \end{aligned} \tag{5.38}$$

It should be clear how to extend this argument to a general $2M$-point function.

If we want to separate in F_{2M} the non-universal and the universal terms, we have to extract from equation (5.31) more detailed knowledge about the 2-point function than that given by the expansion (5.32). Since F_2 starts with an $\mathcal{O}(L^2)$ term, we have to include into the expansion of, for example, F_4 all terms of F_2 down to the order $\mathcal{O}(L^{-2})$. Since

$$F_2(\mathbf{x}) \;=\; \tfrac{1}{2D_1} \sum_{n=0}^{4} \tfrac{1}{n!}\, L^{2-n}\, (\partial_{i_1}\ldots\partial_{i_n}\,(-\Delta)^{-1}\mathcal{C})(0)\; x^{i_1}\cdots x^{i_n} \;+\; \mathcal{O}(L^{-3})\;, \tag{5.39}$$

$$\begin{aligned} F_4(\mathbf{X},\mathbf{Y},\mathbf{Z}) \;=\; & \tfrac{1}{4D_1^2} \sum_{\substack{m,n\geq 0\\ n+m\leq 4}} \tfrac{1}{m!}\,\tfrac{1}{n!}\, L^{4-m-n}\, (\partial_{i_1}\ldots\partial_{i_m}\,\Delta^{-1}\mathcal{C})(0) \\ & \cdot (\partial_{j_1}\ldots\partial_{j_n}\,\Delta^{-1}\mathcal{C})(0)\; X^{i_1}\cdots X^{i_m}\, Z^{j_1}\cdots Z^{j_n} \;+\; \mathcal{O}(L^{-1})\;. \end{aligned} \tag{5.40}$$

For rotationally invariant $\mathcal{C}$, this reduces to

$$\begin{aligned} F_4(\mathbf{X},\mathbf{Y},\mathbf{Z}) \;=\; & \tfrac{1}{4D_1^2}\Big(L^4\,(\Delta^{-1}\mathcal{C})(0)^2 + \tfrac{1}{6}\,L^2\,\mathcal{C}(0)\,(\Delta^{-1}\mathcal{C})(0)\,(|\mathbf{X}|^2+|\mathbf{Z}|^2) \\ & -\tfrac{1}{144}\,(\Delta\mathcal{C})(0)\,(\Delta^{-1}\mathcal{C})(0)\,(|\mathbf{X}|^4+|\mathbf{Z}|^4) + \tfrac{1}{36}\,\mathcal{C}(0)^2\,|\mathbf{X}|^2|\mathbf{Z}|^2\Big) + \mathcal{O}(L^{-2})\;. \end{aligned} \tag{5.41}$$

Note the presence of the universal Gaussian contribution $F_2^{\rm sc}(\mathbf{X})\,F_2^{\rm sc}(\mathbf{Z}) = \frac{\epsilon^2}{36\,D_1^2}\,|\mathbf{X}|^2\,|\mathbf{Z}|^2$ on the right hand side. The rest of the terms are non-universal zero modes of the operator

$$\mathcal{M}_2^{\rm sc}\otimes\mathcal{M}_2^{\rm sc}\,(\mathcal{M}_2^{\rm sc}\otimes 1 + 1\otimes\mathcal{M}_2^{\rm sc})^{-1}\,\mathcal{M}_4^{\rm sc} \tag{5.42}$$

equal to $\mathcal{M}_2^{\rm sc}\otimes\mathcal{M}_2^{\rm sc} = 4D_1^2\,\Delta_{\mathbf{X}}\Delta_{\mathbf{Z}}$ in the action on $\mathbf{Y}$-independent functions. That this must be the case may be easily seen by rewriting the relation (5.34) as the equation

$$(\mathcal{M}_2^{\rm sc}\otimes\mathcal{M}_2^{\rm sc})\,(\mathcal{M}_2^{\rm sc}\otimes 1 + 1\otimes\mathcal{M}_2^{\rm sc})^{-1}\,\mathcal{M}_4^{\rm sc}\;F_4 \;=\; \mathcal{C}_L\otimes\mathcal{C}_L\;. \tag{5.43}$$

Note that the separation of the universal Gaussian contribution $F_2^{\rm sc}(\mathbf{X})\,F_2^{\rm sc}(\mathbf{Z})$ from the terms with $m=n=2$ is ambiguous since there are non-universal terms of the same homogeneity degree. Nevertheless, the calculation for $\kappa=0$ confirms the general structure of the correlations with the non-universal terms given by homogeneous zero modes of differential operators.

5.4 4-point and 6-point function for non-zero κ

Let us consider the 4-point function given by Eqs. (5.33) and (5.34) for general κ between 0 and 2. Recalling that $F_2 = \mathcal{M}_2^{-1}\mathcal{C}_L$, it may be convenient to view Eq. (5.34) as the differential equation for F_4:

$$\mathcal{M}_4\,F_4 \;=\; F_2\otimes\mathcal{C}_L + \mathcal{C}_L\otimes F_2\;. \tag{5.44}$$

The connected part $F_4^c = F_4 - F_2\otimes F_2$ of F_4 satisfies the equation

$$\mathcal{M}_4\,F_4^c \;=\; \mathcal{L}\,(F_2\otimes F_2)\;, \tag{5.45}$$

where $\mathcal{L}\equiv(\mathcal{K})_{1,3}+(\mathcal{K})_{1,4}+(\mathcal{K})_{2,3}+(\mathcal{K})_{2,4}$, see Eq. (4.34).

First, let us study the L-dependence of $F_2\otimes F_2$. Since, as follows from Eq. (5.4) with $\nu=0$,

$$F_2(\mathbf{x}) \;=\; \gamma\,L^{2-\kappa} \,-\, \tfrac{\epsilon}{3(2-\kappa)D_1}\,|\mathbf{x}|^{2-\kappa} \;+\; \mathcal{O}(L^{-2})\;, \tag{5.46}$$

we infer that the product of 2-point functions

$$F_2(\mathbf{X})\, F_2(\mathbf{Z}) \;=\; \gamma^2\, L^{2(2-\kappa)} \;+\; \gamma\, \tfrac{\epsilon}{3\,(2-\kappa)\, D_1}\, L^{2-\kappa}\, (|\mathbf{X}|^{2-\kappa} + |\mathbf{Z}|^{2-\kappa}) \\ +\; \tfrac{\epsilon^2}{9\,(2-\kappa)^2\, D_1^2}\, |\mathbf{X}|^{2-\kappa}\, |\mathbf{Z}|^{2-\kappa} \;+\; \mathcal{O}(L^{-\kappa})\;. \tag{5.47}$$

Compare this expression with the similar expression (5.41) for $\kappa = 0$. The term $\sim (|\mathbf{X}|^4 + |\mathbf{Z}|^4)$ present there comes from the $\mathcal{O}(L^{-\kappa})$ terms of the $\kappa > 0$ case.

Let us return to the connected contribution to F_4 as given by Eq. (5.45). In the action on translationally invariant functions,

$$\begin{aligned} \mathcal{M}_4 \;=\; & -\; \widetilde{D}^{ij}(\mathbf{X})\, \partial_{X^i} \partial_{X^j} \;-\; \widetilde{D}^{ij}(\mathbf{Y})\, \partial_{Y^i} \partial_{Y^j} \;-\; \widetilde{D}^{ij}(\mathbf{Z})\, \partial_{Z^i} \partial_{Z^j} \\ & -\; \Big(\widetilde{D}^{ij}(\mathbf{X}+\mathbf{Y}) - \widetilde{D}^{ij}(\mathbf{X}) - \widetilde{D}^{ij}(\mathbf{Y})\Big)\, \partial_{X^i} \partial_{Y^j} \\ & -\; \Big(\widetilde{D}^{ij}(\mathbf{Y}+\mathbf{Z}) - \widetilde{D}^{ij}(\mathbf{Y}) - \widetilde{D}^{ij}(\mathbf{Z})\Big)\, \partial_{Y^i} \partial_{Z^j} \\ & -\; \Big(\widetilde{D}^{ij}(\mathbf{X}+\mathbf{Y}+\mathbf{Z}) - \widetilde{D}^{ij}(\mathbf{X}+\mathbf{Y}) - \widetilde{D}^{ij}(\mathbf{Y}+\mathbf{Z}) + \widetilde{D}^{ij}(\mathbf{Y})\Big)\, \partial_{X^i} \partial_{Z^j} \end{aligned} \tag{5.48}$$

for $\nu = 0$. Note that

$$\mathcal{L}\,(F_2 \otimes F_2)(\mathbf{X},\mathbf{Y},\mathbf{Z}) \;=\; \Big(\widetilde{D}^{ij}(\mathbf{X}+\mathbf{Y}+\mathbf{Z}) - \widetilde{D}^{ij}(\mathbf{X}+\mathbf{Y}) \\ - \widetilde{D}^{ij}(\mathbf{Y}+\mathbf{Z}) + \widetilde{D}^{ij}(\mathbf{Y})\Big)\, (\partial_i F_2)(\mathbf{X})\, (\partial_j F_2)(\mathbf{Z}) \tag{5.49}$$

has a well defined limit when $L \to \infty$ since it involves only the derivatives of F_2. In particular for $m = 0$ and $\nu = 0$, the $L \to \infty$ limit of $\mathcal{L}\,(F_2 \otimes F_2)$ is equal to

$$\tfrac{\epsilon^2}{9\,(2-\kappa)^2\, D_1^2}\, \mathcal{L}^{\rm sc}\, |\mathbf{X}|^{2-\kappa} |\mathbf{Z}|^{2-\kappa} \tag{5.50}$$

and is a homogeneous (rotationally-invariant) function of $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ of degree $2-\kappa$ whose explicit form may be easily found. It is not difficult to write down a solution $F_4^{c\,{\rm sc}}$ of Eq. (5.45) for the limiting case. Indeed, for

$$F_4^{c\,{\rm sc}} \;=\; \tfrac{\epsilon^2}{6\,(2-\kappa)^2 (5-\kappa)\, D_1^2}\, (\,|\mathbf{X}|^{2(2-\kappa)} + |\mathbf{Z}|^{2(2-\kappa)}) \;-\; \tfrac{\epsilon^2}{9\,(2-\kappa)^2\, D_1^2}\, |\mathbf{X}|^{2-\kappa} |\mathbf{Z}|^{2-\kappa}\;, \tag{5.51}$$

one obtains with the use of the decomposition $\mathcal{M}_4^{\rm sc} \;=\; \mathcal{M}_2^{\rm sc} \otimes 1 \;+\; 1 \otimes \mathcal{M}_2^{\rm sc} \;-\; \mathcal{L}^{\rm sc}$ the required relation

$$(\mathcal{M}_4^{\rm sc}\, F_4^{\rm sc})(\mathbf{X},\mathbf{Y},\mathbf{Z}) \;=\; \tfrac{\epsilon^2}{9\,(2-\kappa)^2\, D_1^2}\, \mathcal{L}^{\rm sc}\, |\mathbf{X}|^{2-\kappa} |\mathbf{Z}|^{2-\kappa} \tag{5.52}$$

since $\mathcal{M}_2^{\rm sc} \otimes 1 + 1 \otimes \mathcal{M}_2^{\rm sc}$ does not contribute to the left hand side and $\mathcal{L}^{\rm sc}$ annihilates functions depending only on $\mathbf{X}$ or only on $\mathbf{Z}$. By the same argument as for the two-point function, the solution for finite but large L should differ from the universal scaling form by zero modes of $\mathcal{M}_4^{\rm sc}$ so that

$$F_4^c \;-\; \sum_{0 \le \zeta_{4,n} \le 2(2-\kappa)} L^{2(2-\kappa)-\zeta_{4,n}} \sum_m \gamma_{nm}\, F^c_{4,nm} \;\; \underset{L\to\infty}{\longrightarrow} \;\; F_4^{c\,{\rm sc}} \tag{5.53}$$

where $F^c_{4,nm}$ are homogeneous zero modes of $\mathcal{M}_4^{\rm sc}$ of degree $\zeta_{4,n}$ and the non-universal coefficients γ_{nm} depend on the source covariance $\mathcal{C}$. In fact, a more complicated behavior with logarithmic terms

$$F_4^c \;-\; \sum_{0 \le \zeta_{4,n} \le 2(2-\kappa)} L^{2(2-\kappa)-\zeta_{4,n}} \sum_{m,k} \gamma_{nmk}\, (\ln\,(L f_{nm}))^k\, F^c_{4,nm} \;\; \underset{L\to\infty}{\longrightarrow} \;\; F_4^{c\,{\rm sc}} \tag{5.54}$$

is possible if $\mathcal{M}_4$ has zero modes $(\ln f_{nm})^l\, F_{nm}\,,\ l \leq k\,,$ for f_{nm} homogeneous functions of degree -1.

The main problem which remains in the analysis of the universality of the 4-point functions is to find homogeneous zero modes of $\mathcal{M}_4^{\rm sc}$ with degree between zero and $2(2-\kappa)$ and to decide which ones appear in the relation (5.53). The study of such zero modes seems to lie beyond the scope of the techniques [27] developed for the singular elliptic operators of the type we encounter here. Note that in the full connected 4-point function only the zero modes with permutation symmetry survive. If the sources are not only homogeneous but also isotropic, only $SO(3)$-symmetric zero modes may appear. In fact, one should expect that such zero modes do enter into the expansions (5.53) or (5.54) for generic $\mathcal{C}$ whenever not forbidden by symmetries (which is a somewhat tautological statement). Existence of the zero-mode contributions to the 4-point function is important from the practical point of view. If we study a combination of 4-point functions (e.g. the $4^{\rm th}$ structure function) and we fix the range of variation of distances between the points, then the non-universal terms with the smallest $\zeta_{4,n}^{\rm min} < 2(2-\kappa)$ dominate the behavior for large L and we would see the anomalous exponent $\zeta_{4,n}^{\rm min}$ rather than $2(2-\kappa)$ as governing the distance behavior. The possible appearance of zero modes of $\mathcal{M}_4^{\rm sc}$ in the asymptotic behavior of the 4-point function would lead to the picture of **restricted universality**, holding strictly only for the 4-point correlator **infrared-renormalized** by subtracting homogeneous terms with non-universal coefficients typically divergent when the integral scale L grows to infinity. In the presence of non-universal terms with $\zeta_{4,n} = 2(2-\kappa)$ with are L-independent, the separation between the universal and non-universal terms in the 4-point function would become ambiguous and the Eq. (5.51) would involve an arbitrary choice.

Notice that adding the connected and disconnected contributions to F_4, one obtains (in the absence of logarithmic terms)

$$F_4(\mathbf{X},\mathbf{Y},\mathbf{Z}) \,=\, G(\mathbf{X}) + G(\mathbf{Z}) + \sum_{\zeta_{4,n}} L^{2(2-\kappa)-\zeta_{4,n}} \sum_m \gamma_{nm}\, F^c_{4,nm}(\mathbf{X},\mathbf{Y},\mathbf{Z}) \,+\, o(L) \qquad (5.55)$$

for $G(\mathbf{x}) \,=\, \frac{1}{2}\gamma^2\, L^{2(2-\kappa)} \,+\, \gamma\, \frac{\epsilon}{3(2-\kappa)\,D_1}\, L^{2-\kappa}\, |\mathbf{x}|^{2-\kappa} \,+\, \frac{\epsilon^2}{12\,(2-\kappa)^2\,(5-\kappa)\,D_1^2}\, |\mathbf{x}|^{2(2-\kappa)}$. Only $F^c_{4,mn}$ terms may contribute to the structure function $S_4(r)$, as all the terms dependent on only one variable difference do not survive in S_4. Hence the presence of the non-trivial contributions from the (generally) non-universal terms $F^c_{4,nm}$ is a necessary condition for the very non-triviality of S_4 for large L.

The 6 point function may be treated similarly. The connected part F_6^c of the single 2-particle channel contribution F_6 is given by

$$\begin{aligned} F_6(\mathbf{x}_1,\dots,\mathbf{x}_6) \,&=\, F_6^c(\mathbf{x}_1,\dots,\mathbf{x}_6) \,+\, F_4^c(\mathbf{x}_1,\dots,\mathbf{x}_4)\, F_2(\mathbf{x}_5,\mathbf{x}_6) \\ &+\, F_4^c(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_5\mathbf{x}_6)\, F_2(\mathbf{x}_3,\mathbf{x}_4) \,+\, F_4^c(\mathbf{x}_3,\dots,\mathbf{x}_6)\, F_2(\mathbf{x}_1,\mathbf{x}_2) \\ &+\, F_2(\mathbf{x}_1,\mathbf{x}_2)\, F_2(\mathbf{x}_3,\mathbf{x}_4)\, F_2(\mathbf{x}_5,\mathbf{x}_6)\,, \end{aligned} \qquad (5.56)$$

see Eq. (4.31). It satisfies Eq. (4.34) which reads

$$\begin{aligned} \mathcal{M}_6\, F_6^c \,=\, & ((\mathcal{L}_{1,3} \,+\, \mathcal{L}_{2,3})\,(F_4^c(1,2)\, F_2(3)) \,+\, (\mathcal{L}_{1,2} \,+\, \mathcal{L}_{2,3})\,((F_4^c(1,3)\, F_2(2)) \\ & +(\mathcal{L}_{1,2} \,+\, \mathcal{L}_{1,3})\,(F_2(1)\, F_4^c(2,3))\,, \end{aligned} \qquad (5.57)$$

where the numbers in the paranthesis label the 3 consecutive pairs of points. In the case that the non-universal non-constant zero modes do not appear in the connected 4-point

function, the right hand side of Eq. (5.57) has a finite $L\to\infty$ limit so that the relation of the type (5.53),

$$F_6^c \;-\; \sum_{0\le\zeta_{6,n}\le 3(2-\kappa)} L^{3(2-\kappa)-\zeta_{6,n}}\sum_m \gamma_{nm}\,F_{6,nm}^c \quad\mathop{\longrightarrow}_{L\to\infty}\quad F_6^{c\,\mathrm{sc}}\,, \tag{5.58}$$

with $F_{6,mn}^c$ being homogeneous zero modes of the degree $\zeta_{6,n}$ of $\mathcal{M}_6^{\mathrm{sc}}$ should hold for $m=0=\nu$ (with eventual additional logarithmic terms, as in (5.54)). As we shall see in the next Section such zero modes with rotational and permutation symmetries exist at least for small κ. Hence, the non-constant non-universal terms should appear if not already in the connected 4-point function then in the connected 6-point function. Whether they are visible in the corresponding structure functions is a more difficult question to which we do not have definite answer yet.

Let us end this section by stressing a difference between the infrared renormalization required by the correlation functions of the PS and the renormalization of infrared divergences occurring in massless models of statistical mechanics or field theory. There the divergences are an artifact of the perturbative expansion and signal that the control parameters of the system are off their critical values. The divergences may be removed by finite shifts of the control parameters which depend non-smoothly on the coupling constants (hence the infinities in the perturbative treatment). On the contrary, in the PS model of turbulent advection, the genuine divergences appear when the size of the system tends to infinity and the correlators become finite only after the subtraction of the diverging terms. Similar picture appeared in Polyakov's 2-dimensional conformal turbulence [20] where it was argued that extracting the conformal-invariant part of the correlation functions requires subtraction of non-universal polynomial terms in the position space[5]. We expect the infrared renormalization of the type described above to be the general feature of universality in turbulent systems and the most important lesson to learn from the study of the PS even if, at present, we are unable to decide whether non-universal terms are present in the structure functions, the most commonly studied correlators.

5.5 κ-expansion

We may study the homogeneous zero modes of the operator $\mathcal{M}_4^{\mathrm{sc}}$ in perturbation expansion in powers of κ. We shall work here only to the first order in κ. Eq. (4.6) implies that (for $m=0$)

$$\begin{aligned}\widetilde{D}^{ij}(\mathbf{x}) \;&=\; 2D_1\,\delta^{ij} \;+\; \kappa\,D_1\,\Big(2\,\delta^{ij}\,\ln|\mathbf{x}| \;-\; x^ix^j\,|\mathbf{x}|^{-2} \;+\; \delta^{ij}\Big) \;+\; \mathcal{O}(\kappa^2)\\ &\equiv\; 2D_1\,\delta^{ij} \;+\; 2\kappa\,D_1\,R^{ij}(\mathbf{x}) \;+\; \mathcal{O}(\kappa^2)\;.\end{aligned} \tag{5.59}$$

As a result, to the first order in κ,

$$\begin{aligned}\mathcal{M}_4^{\mathrm{sc}} \;=&\; \mathcal{M}_{4,0}^{\mathrm{sc}} \;-\; 2\kappa\,D_1\,R^{ij}(\mathbf{X})\,\partial_{X^i}\partial_{X^j} \;-\; 2\kappa\,D_1\,R^{ij}(\mathbf{Y})\,\partial_{Y^i}\partial_{Y^j} \;-\; 2\kappa\,D_1\,R^{ij}(\mathbf{Z})\,\partial_{Z^i}\partial_{Z^j}\\ &-\; 2\kappa\,D_1\,\Big(R^{ij}(\mathbf{X}+\mathbf{Y}) - R^{ij}(\mathbf{X}) - R^{ij}(\mathbf{Y})\Big)\,\partial_{X^i}\partial_{Y^j}\\ &-\; 2\kappa\,D_1\,\Big(R^{ij}(\mathbf{Y}+\mathbf{Z}) - R^{ij}(\mathbf{Y}) - R^{ij}(\mathbf{Z})\Big)\,\partial_{Y^i}\partial_{Z^j}\\ &-\; 2\kappa\,D_1\,\Big(R^{ij}(\mathbf{X}+\mathbf{Y}+\mathbf{Z}) - R^{ij}(\mathbf{X}+\mathbf{Y}) - R^{ij}(\mathbf{Y}+\mathbf{Z}) + R^{ij}(\mathbf{Y})\Big)\,\partial_{X^i}\partial_{Z^j}\\ &\equiv\; \mathcal{M}_{4,0}^{\mathrm{sc}} \;+\; 2\kappa\,D_1\,V_4\;,\end{aligned} \tag{5.60}$$

[5] K.G. thanks A. Polyakov for discussion of this point

where the subscript "0" refers to $\kappa = 0$. $\mathcal{M}_4^{\rm sc}$ commutes with 3-dimensional translations and rotations and with the permutations of four points. We shall search for its zero modes respecting these symmetries. Note that

$$\mathcal{M}_{4,0}^{\rm sc} \;=\; -2D_1\,(\Delta_{\widetilde{\mathbf{X}}} + \Delta_{\widetilde{\mathbf{Y}}} + \Delta_{\widetilde{\mathbf{Z}}})\;, \tag{5.61}$$

where

$$\widetilde{\mathbf{X}} \,=\, \mathbf{X}\,, \quad \widetilde{\mathbf{Y}} \,=\, \sqrt{2}(\mathbf{Y} + \tfrac{1}{2}\mathbf{X} + \tfrac{1}{2}\mathbf{Z})\,, \quad \widetilde{\mathbf{Z}} \,=\, \mathbf{Z}\;. \tag{5.62}$$

Denoting $R \equiv (\widetilde{\mathbf{X}}^2 + \widetilde{\mathbf{Y}}^2 + \widetilde{\mathbf{Z}}^2)^{1/2}$, we obtain

$$\mathcal{M}_{4,0}^{\rm sc} \;=\; -\tfrac{2D_1}{R^8}\,\partial_R\, R^8\,\partial_R \,+\, \tfrac{2D_1}{R^2}\,\Phi \tag{5.63}$$

where Φ is the Laplacian on the sphere S^8 in the space of $(\widetilde{\mathbf{X}},\widetilde{\mathbf{Y}},\widetilde{\mathbf{Z}})$. The symmetric zero modes of the lowest homogeneity of $\mathcal{M}_{4,0}^{\rm sc}$ occur in degree zero (constants) and in degree 4. The latter, when expressed in terms of point-differences, have the form

$$a\sum_{\{n,n'\}}(\mathbf{x}_n - \mathbf{x}_{n'})^4 \,+\, b\sum_{\{\{n,m\},\{n,m'\}\}}(\mathbf{x}_n - \mathbf{x}_m)^2\,(\mathbf{x}_n - \mathbf{x}_{m'})^2 \,+\, c\sum_{\{\{n,n'\},\{m,m'\}\}}(\mathbf{x}_n - \mathbf{x}_{n'})^2\,(\mathbf{x}_m - \mathbf{x}_{m'})^2\,, \tag{5.64}$$

where the pairs $\{n,n'\}$ and $\{m,m'\}$ are assumed different, as well as the pairs $\{n,m\}$ and $\{n,m'\}$ and where

$$20a + 28b + 6c = 0\;. \tag{5.65}$$

The constant survives as the eigenvalue of $\mathcal{M}_4^{\rm sc}$ for $\kappa \neq 0$. The change of the two independent zero modes $R^4 f_i$, $i = 1,2$, of the form (5.64) may be traced by the standard degenerate perturbation theory. This is done as follows. Looking for a homogeneous zero mode $R^{4+\kappa\lambda}\,(a_1 f_1 + a_2 f_2 + \kappa\, f_3)$ with a homogeneous degree zero function f_3 orthogonal to $f_{1,2}$ in $L^2(S^8)$, we obtain in the linear order in κ

$$\mathcal{M}_{4,0}^{\rm sc}\,\Big(\lambda\, R^4\,\ln R\,(a_1 f_1 + a_2 f_2) \,+\, R^4\, f_3\Big) \,+\, 2D_1\, V_4\, R^4\,(a_1 f_1 + a_2 f_2) \;=\; 0 \tag{5.66}$$

or, using the form (5.63) of $\mathcal{M}_{4,0}^{\rm sc}$,

$$-\,15\,\lambda\,(a_1 f_1 + a_2 f_2) \,-\, 44\, f_3 \,+\, \Phi\, f_3 \,+\, \tfrac{1}{R^2}\, V_4\, R^4\,(a_1 f_1 + a_2 f_2) \;=\; 0\;. \tag{5.67}$$

Upon taking the $L^2(S^8)$ scalar products with $f_{1,2}$, f_3 drops out resulting in the relation

$$-\,15\,\lambda \sum_{j=1,2}(f_i,\, f_j)\, a_j \,+\, \sum_{j=1,2}(f_i,\, \tfrac{1}{R^2}\, V_4\, R^4\, f_j)\, a_j \;=\; 0 \tag{5.68}$$

Hence λ has to solve the equation

$$\det\Big((f_i,\, \tfrac{1}{15\,R^2}\, V_4\, R^4\, f_j) \,-\, \lambda\,(f_i,\, f_j)\Big) \;=\; 0\;. \tag{5.69}$$

It may be decided numerically whether this equation has solutions $\lambda < -2$ so that there exist homogeneous zero mode of $\mathcal{M}_4^{\rm sc}$ with degree $\leq 4 - 2\kappa$ for small κ which, multiplied by $\gamma\, L^{-(2+\lambda)\kappa}$, should appear as non-universal terms in the truncated 4-point function. Given λ and a non-zero solution (a_1, a_2) of Eq. (5.68), the $\mathcal{O}(\kappa)$ contribution f_3 to the zero mode should be computed from Eq. (5.67).

If Eq. (5.69) has one double root but there is only one solution (a_1, a_2) (up to normalization) of Eq. (5.68) then one should look for an additional zero mode of $\mathcal{M}_4^{\rm sc}$ of the form

$$\kappa\,(\ln R)\,R^{4+\kappa\lambda}\,(a_1\,f_1\,+\,a_2\,f_2\,+\,\kappa\,f_3)\,+\,R^{4+\kappa\lambda}\,(b_1\,f_1\,+\,b_2\,f_2\,+\,\kappa\,f_4)\;. \tag{5.70}$$

This leads in the first order to the relation

$$-\,15\,((a_1+\lambda b_1)f_1+(a_2+\lambda b_2)f_2)\,-\,44\,f_4\,+\,\Phi\,f_4\,+\,\tfrac{1}{R^2}\,V_4\,R^4\,(b_1f_1+b_2f_2)\,=\,0 \tag{5.71}$$

which implies that

$$-\,15\sum_{j=1,2}(f_i,\,f_j)\,(a_j+\lambda\,b_j)\,+\,\sum_{j=1,2}(f_i,\,\tfrac{1}{R^2}\,V_4\,R^4\,f_j)\,b_j\,=\,0\;. \tag{5.72}$$

The latter equation has a non-zero solution for (b_1, b_2). In this case, the logarithmic terms $\sim\,(\ln L)\,L^{-(2+\lambda)\kappa}$ should appear in the asymptotic expansion of F_4^c for $\kappa > 0$ if $\lambda \le -2$.

The zero modes of the scaling operator $\mathcal{M}_6^{\rm sc}$ may be studied similarly by writing

$$\mathcal{M}_6^{\rm sc}\,=\,\mathcal{M}_{6,0}^{\rm sc}\,+2\,\kappa\,D_1\,V_6\;, \tag{5.73}$$

and using the perturbation expansion. By a linear change of variables, $\mathcal{M}_{6,0}^{\rm sc}$ is equivalent to the Laplacian in $\mathbf{R}^{15}$. As for $\mathcal{M}_{4,0}^{\rm sc}$, the homogeneous zero modes of $\mathcal{M}_{6,0}^{\rm sc}$ with the lowest degree, symmetric under rotations and permutations, are constants and the $4^{\rm th}$-order polynomials (5.64) satisfying this time the relation

$$20a + 56b + 36c = 0\;. \tag{5.74}$$

For $\kappa > 0$ the constants are still annihilated by $\mathcal{M}_{6,0}^{\rm sc}$ whereas the two $4^{\rm th}$-order zero modes may change slightly the homogeneity degree (one of them may also pick a logarithm). As for small κ the deformed degree will still be smaller than $3(2-\kappa)$, the deformed zero modes, accompanied by a positive power of L and a non-universal coefficient, should appear in the asymptotic expansion of the connected 6-point function discussed above.

6 Conclusions

At the beginning of these lectures, we have sketched the main idea of Kolmogorov's picture of universality in the fully developed turbulence, related to the energy cascade. We briefly described how the turbulence problem may be formulated in the language formally very close to that of equilibrium statistical mechanics or Euclidean field theory stressing, however, the important differences between the two classes of problems. The bulk of the lectures was devoted to the discussion of a simple model of turbulent phenomenon, the passive scalar (PS), which we treated as a test ground for the universality ideas. The exact solution for the model in terms of singular elliptic many-body operators showed non-universal diffusive decay of correlations for deterministic initial data but a universal super-diffusive decay for random homogeneous initial data. In the presence of random sources injecting the energy at small wavenumbers, the steady state exhibited an inertial range with the energy cascade and universal scaling in the connected equal-time two-point function of the scalar. We have argued that the higher-point equal-time connected

functions are expected to show only restricted universality with the terms dominating for large size systems characterized by universal anomalous exponents and non-universal amplitudes. We have shown that such terms should appear in the 6-point function if not in the 4-point one, at least for small velocity exponent κ. Further work along well traced direction is required to decide on this question as well as on the question what terms dominate the respective structure functions. One of the open problems is to relate the terms with anomalous exponents to the physical idea of intermittency. One should also understand what role, if any, is played in the model by the volume preserving diffeomorphisms. Upon infrared renormalization which subtracts the non-universal terms, the equal-time correlators of the scalar should become finite and universal (with normal scaling) in the limit when molecular diffusivity $\nu \to 0$ and the integral scale $L \to \infty$. In particular, no ν-dependent subtractions are needed in the steady-state equal-time correlators of the scalar and their short-distance behavior is easy to analyze [14][15]. This behavior determines what ultraviolet (short distance) renormalization is required by the composite operators involving the gradients of T. The basic role of ν is to assure the convergence to the steady state by providing the mechanism which removes on small scales the energy injected by the sources at large distances.

We expect the main features of the behavior of the PS described above to be present also in other systems exhibiting a fully developed turbulence, including the Navier-Stokes flow. Still much remains to be done even for the PS. One important step could be a complete perturbative analysis with the above picture of restricted universality established order by order. The κ expansion discussed above is a possibility but already the analysis of the lowest terms proved to be quite complicated [26]. Other possibilities are the quasi-Lagrangian perturbation expansion or the expansion proposed in [14]. Ultimately, if no way to compute exactly the non-universal terms in the correlators is found, a multiscale analysis employing perturbative arguments in conjunction with a renormalization group type analysis should provide a tool to fully control the behavior of correlation functions. Developing such methods for the PS may be a first step towards a complete theory of fully developed turbulence. More than half-century after Kolmogorov's work, despite further progress, a fully developed understanding of the fully developed turbulence remains a major challenge.

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